Kumamoto Journal of Science

Series A (Mathematics, Physics and Chemistry)

Vol. V, 1960 — 1962

第一部 第五巻

総 目 次

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熊本大学理学部紀要

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CONTENTS

THE POINT ESTIMATION OF THE PARAMETERS IN THE MIXED MODEL

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(Received July 31, 1960)

1. Introduction.

In this paper we shall be concerned with the estimation of the parameters in the mixed model, which is a sort of continuation of the previous work of the author [2][1] concerning the estimation of the variance components of the r-way layout of random effect model.

The mixed model is understood to be the model of the r-way layout where some of the treatment effects are the normal variables and the effect of the remaining treatments are constants. In this paper we assume the means of the random effects are all zero, which should not be restrictive at all for the problem of estimation, and we shall be concerned with the estimation of the variances of the random effects, the constant effects and the general mean.

For the development of our arguments, we need to define two types of mixed models. Stating the model in more details, we shall take all the interactions up till the highest order in our consideration, and it seems to be reasonable to assume that all the interaction effects between the fixed main effects are constant and all the other type of interactions are random. Namely if an interaction involves at least a factor whose main effect is random, then this interaction effect is also assumed to be random and on the other hand if an interaction involves no factor whose main effect is random, then this interaction effect is assumed to be constant.

Under snch assumptions as stated above, the two types of mixed models are defined from merely a technical reason. Type I is the mixed model involving only one random main effect, whereas Type II is the one involving more than one random main effect.

The results obtained in this paper are the derivation of the joint density function of all observations in the r-way layout of the mixed model, and the proof of the completeness of the family of distributions of the sufficient statistics in our concern, which implies by making use of the result due to Lehmann and Scheffé that the estimates of the parameters which are usually adopted in the practice of statistical inferences as unbiased estimates are the unique minimum variance unbiased estimates. In Section 2, we shall define some notations similar to those in the previous paper of the author [2] and give the model equation in our concern. Section 3 is devoted to Type I model and Section 4 is to Type II model. In Section 5, we shall remark that the result in the previous paper of the author can be improved by making use of a result of this paper.

1) Numbers in brackets refer to the references of the end of the paper.

2. Preliminaries.

We shall be concerned with the r-way layout of the mixed model whose model equation is given by the following

$$(2.1)\qquad x_{t_0t_1\cdots t_r}=\mu+\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\alpha(j_1,\cdots,j_h;t_{j_1},\cdots,t_{j_h})$$
$$+\sum_{h=0}^{r-s}\sum_{J_h\subset R-S}\sum_{k=1}^{s}\sum_{I_k\subset S}a(i_1,\cdots,i_k,j_1,\cdots,j_h;t_{i_1},\cdots,t_{i_k},t_{j_1},\cdots,t_{j_h}),$$
$$(t_{i_c}=1,2,\cdots,n_{i_c};t_{j_d}=1,2,\cdots,n_{j_d};t_0=1,2,\cdots,n_0;c=1,\cdots,s;d=1,\cdots,r-s),$$

where μ denotes the general mean, $\alpha(j_1,\cdots,j_h;t_{j_1},\cdots,t_{j_h})$ denotes the interaction between j_1-th, j_2-th, $\cdots$, j_h-th factors with the level $t_{j_1}, t_{j_2},\cdots,t_{j_h}$, $a(i_1,\cdots,i_k,j_1,\cdots,j_h;t_{i_1},\cdots,t_{i_k},t_{j_1},\cdots,t_{j_h})$ denotes the interaction between i_1-th, i_2-th, $\cdots$, i_k-th, j_1-th, j_2-th, $\cdots$, j_h-th factors with the level $t_{i_1},\cdots t_{i_k},t_{j_1},\cdots t_{j_h}$, and $e_{t_0t_1\cdots t_r}$ denotes the error term. When $h=0$, $a(i_1,\cdots,i_k,j_1,\cdots,j_h;t_{i_1},\cdots,t_{i_k},t_{j_1},\cdots,t_{j_h})$ is understood to be $a(i_1,\cdots,i_k;t_{i_1},\cdots t_{i_k})$. In the above equation S and $R-S$ denote the sets of integers $(1,2,\cdots,s)$ and $(s+1,s+2,\cdots,r)$ respectively. I_k and J_h denote the subsets $(i_1,i_2,\cdots,i_k)$ of $S=(1,2,\cdots,s)$ and $(j_1,j_2,\cdots,j_h)$ of $R-S=(s+1,s+2,\cdots,r)$, with the relations $i_1<i_2<\cdots<i_k$ and $j_1<j_2<\cdots<j_h$ respectively. $\sum_{I_k\subset S}$ and $\sum_{J_h\subset R-S}$ denote the summations for all subsets I_k of size k in S and for all subsets J_h of size h in $R-S$ respectively.

We assume that μ and all $\alpha(j_1,\cdots,j_h;t_{j_1},\cdots,t_{j_h})$ are constants, all $a(i_1,\cdots,i_k,j_1,\cdots,j_h;t_{i_1},\cdots,t_{i_k},t_{j_1},\cdots,t_{j_h})$ and $e_{t_0t_1\cdots t_r}$ are distributed independently to each other in normal distributions with mean all equal to zero and the variance of each $a(i_1,\cdots,i_k,j_1,\cdots,j_h;t_{i_1},\cdots,t_{i_k},t_{j_1},\cdots,t_{j_h})$ equals to $\sigma^2_{I_kJ_h}$, the variance of each $a(i_1,\cdots,i_k;t_{i_1},\cdots,t_{i_k})$ equals to $\sigma^2_{I_k}$ and the variances of $e_{t_0t_1\cdots t_r}$ all equal to σ^2_0.

Further we assume that there hold relations:

$$(2.2)\qquad \sum_{t_{j_c}=1}^{n_{j_c}}\alpha(j_1,\cdots,j_h;t_{j_1},\cdots,t_{j_h})=0,$$
$$(J_h\subset R-S;c=1,2,\cdots,h).$$

The above stated is the general formulation of the mixed model under general assumptions, and we shall define two types of them as follows. We call the mixed model, whose model equation is represented by (2.1) for $s=1$, satisfying all assumptions stated above, Type I model and the one for $s\geqq 2$ Type II model.

Throughout this paper the notations such as U_d, V_e, M_α and N_β etc. mean the sets of integers $(u_1,u_2,\cdots,u_d)$, $(v_1,v_2,\cdots,v_e)$, $(m_1,m_2,\cdots,m_\alpha)$ and $(n_1,n_2,\cdots,n_\beta)$ etc., while these mean the empty set if d,e and α etc. equal to zero, and R,S,T etc. mean the set of integers $(1,2,\cdots,r)$, $(1,2,\cdots,s)$, $(1,2,\cdots,t)$ etc. and on these sets of integers we shall make use of the set theoretical notations such as $A\cup B, A\cap B, A\subset B, A-B$ etc. Further, we assume that the summations such as $\sum_{A\subset B}a_A$, $\sum_{\substack{A\subset B\\ A\supset C}}a_A$, where A,B,C are such sets of integers as defined above, mean the sum of all numbers a_A's having A as the suffixes which are included in B, and included in B and including C, respectively.

The Kronecker product of two or any number of matrixes are defined in this paper in the way reverse to the usual ones for the convenience in handling the cumbersome notation systems.

Let $A=(a_{ij})$, $B=(b_{ij})$, the Kronecker product denoted by $A\otimes B$ is defined as the matrix (Ab_{ij}), The Kronecker product of any number of matrixes is defined as the natural generalization of two matrixes, we shall write the Kronecker product of n matrixes $A_1, A_2, \cdots, A_n$, as $\prod_{i=1}^{n} \otimes A_i$.

In this paper, we shall make use of the well-known relations concerning the Kronecker products of two matrixes such as $(A\otimes B)(C\otimes D)=AC\otimes BD$, $(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$, $(A\otimes B)'=A'\otimes B'$, and their generalizations to the products of any number of matrixes without mentioning explicitly. Throughout this paper we shall write $n\times n$ unit matrix as I_n, E_n denotes the $n\times n$ matrix with the elements all equal to 1. Let H_n be the $n\times n$ matrix with the elements all equal to zero except for the element of the first row in the first column equal to 1, and let $K_n=I_n-H_n$.

Further let T_n be defined as the orthogonal matrix with the elements of the first column all equal to $\frac{1}{\sqrt{n}}$.

Then we have easily

$$T'_n E_n T_n = nH_n. \tag{2.3}$$

3. The case of Type I.

3. 1. Determinant of the variance matrix.

In this type the model equation is given by

$$\begin{aligned} x_{t_0 t_1 \cdots t_r} = \mu &+ \sum_{h=1}^{r-1} \sum_{J_h \subset R-1} \alpha(j_1, \cdots, j_h; t_{j_1}, \cdots, t_{j_h}) + \sum_{h=0}^{r-1} \sum_{J_h \subset R-1} a(1, j_1, \cdots, j_h; t_1, t_{j_1}, \cdots, t_{j_h}) \\ &+ e_{t_0 t_1 \cdots t_r}, \qquad \begin{pmatrix} t_{j_c}=1, \cdots, n_{j_c}; & c=1, \cdots, s; \\ t_0=1, \cdots, n_0; & t_1=1, \cdots, n_1. \end{pmatrix}. \end{aligned} \tag{3.1}$$

Thus we have the expression of the variance matrix in terms of the Kronecker products as follows,

$$\begin{aligned} V = \sigma_1^2 E_{n_0} \otimes \prod_{\zeta=1}^{r} \otimes \left(E_{n_\zeta}^{1-\delta_1^\zeta} \times I_{n_\zeta}^{\delta_1^\zeta}\right) &+ \sum_{h=1}^{r-1} \sum_{J_h \subset R-1} \sigma_{1J_h}^2 E_{n_0} \otimes \prod_{\zeta=1}^{r} \otimes \left(E_{n_\zeta}^{1-\delta_{1J_h}^\zeta} \times I_{n_\zeta}^{\delta_{1J_h}^\zeta}\right) \\ &+ \sigma^2 I_{n_0} \otimes I_{n_1} \otimes \cdots \otimes I_{n_r}, \end{aligned} \tag{3.2}$$

where $\delta_{M_\alpha}^\zeta$ is a sort of generalization of the Kronecker's delta which is

$$\delta_{M_\alpha}^\zeta = \begin{cases} 1 & \text{if } \zeta \text{ is equal to either of the elements in } M_\alpha, \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

and E^0 of a matrix E is defined to be unit matrix I.

For the development of the arguments in this section we have to define a number

of notations as follows.

DEFINITION 3.1.

$$A_{(1)}B_{(J_h)} \equiv \sum_{\beta=h}^{r-s} \sum_{\substack{N_\beta \supset J_h \\ N_\beta \subset R-1}} \sigma_{1N_\beta}^2 \prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1.N_\beta}^\zeta}, \tag{3.4}$$

$$A_{(1)}B_{(J_h)}^{(V_e)} \equiv \sum_{\beta=h}^{r-s-e} \sum_{\substack{N_\beta \supset J_h \\ N_\beta \subset R-1-V_e}} \sigma_{1N_\beta}^2 \prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1N_\beta V_e}^\zeta}, \tag{3.5}$$

$$A_{(1)}B \equiv \sum_{\beta=0}^{r-s} \sum_{N_\beta \subset R-1} \sigma_{1.N_\beta}^2 \prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1N_\beta}^\zeta}, \tag{3.6}$$

$$C_{(1)}D_{(J_h)} \equiv A_{(1)}B_{(J_h)} + \sigma_0^2, \tag{3.7}$$

$$C_{(1)}D \equiv A_{(1)}B + \sigma_0^2, \tag{3.8}$$

where we assume $\sigma_{1N_\beta}^2 = \sigma_1^2$ when $\beta=0$.

Now with the aid of the above notations we may evaluate the determinant of the variance matrix (3.2).

THEOREM 3.1. *The determinant* $|V|$ *of the variance matrix* V *of* (3.2) *is given in the notations of* (3.7) *and* (3.8) *as follows,*

$$|V| = \prod_{h=0}^{r-1} \prod_{J_h \subset R-1} \{C_{(1)}D_{(J_h)}\}^{n_1(n_{j_1}-1)\cdots n(j_h-1)} \{\sigma_0^2\}^{(n_0-1)n_1 n_2 \cdots n_r}. \tag{3.9}$$

PROOF. Let us at first transform this matrix by the orthogonal matrix which is the Kronecker product of the matrixes T_{n_i} defined in Section 2, and we have

$$\begin{aligned}
(T_{n_0}\otimes T_{n_1}\otimes\cdots\otimes T_{n_r})'V(T_{n_0}\otimes T_{n_1}\otimes\cdots\otimes T_{n_r}) &= \sigma_1^2 \prod_{j=0}^{r} n_j^{1-\delta_1^j} H_{n_0} \otimes \prod_{\zeta=1}^{r} \otimes (H_{n_\zeta}^{1-\delta_1^\zeta} \times I_{n_\zeta}^{\delta_1^\zeta}) \\
&+ \sum_{h=1}^{r-1} \sum_{J_h \subset R-1} \sigma_{1J_h}^2 \prod_{j=0}^{r} n_j^{1-\delta_{1J_h}^j} H_{n_0} \otimes \prod_{\zeta=1}^{r} \otimes (H_{n_\zeta}^{1-\delta_{1J_h}^\zeta} \times I_{n_\zeta}^{\delta_{1J_h}^\zeta}) + \sigma_0^2 I_{n_0} \otimes I_{n_1} \otimes \cdots \otimes I_{n_r} \\
&= \sigma_1^2 \prod_{j=0}^{r} n_j^{1-\delta_1^j} H_{n_0} \otimes \prod_{\zeta=1}^{r} \otimes \left(H_{n_\zeta}^{1-\delta_1^\zeta} \times (H_{n_\zeta}+K_{n_\zeta})^{\delta_1^\zeta}\right) \\
&+ \sum_{h=1}^{r-1} \sum_{J_h \subset R-1} \sigma_{1J_h}^2 \prod_{j=0}^{r} n_j^{1-\delta_{1J_h}^j} H_{n_0} \otimes \prod_{\zeta=1}^{r} \otimes \left(H_{n_\zeta}^{1-\delta_{1J_h}^\zeta} \times (H_{n_\zeta}+K_{n_\zeta})^{\delta_{1J_h}^\zeta}\right) \\
&+ \sigma_0^2 \prod_{\zeta=0}^{r} \otimes (H_{n_\zeta}+K_{n_\zeta}) \\
&= \sigma_1^2 \prod_{j=0}^{r} n_j^{1-\delta_1^j} H_{n_0} \otimes \prod_{\zeta=1}^{r} \otimes (H_{n_\zeta} + \delta_1^\zeta K_{n_\zeta}) \\
&+ \sum_{h=1}^{r-1} \sum_{J_h \subset R-1} \sigma_{1J_h}^2 \prod_{j=0}^{r} n_j^{1-\delta_{1J_h}^j} H_{n_0} \otimes \prod_{\zeta=1}^{r} \otimes (H_{n_\zeta} + \delta_{1J_h}^\zeta K_{n_\zeta}) + \sigma_0^2 \prod_{\zeta=0}^{r} \otimes (H_{n_\zeta} + K_{n_\zeta}).
\end{aligned} \tag{3.10}$$

Then we have

$$|V| = \prod_{h=0}^{r-1} \prod_{J_h \subset R-1} \left\{ \sum_{\beta=h}^{r} \sum_{N\beta \supset J_h} \sigma_{1N\beta}^2 \prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1N\beta}^{\zeta}} + \sigma_0^2 \right\}^{n_1(n_{j_1}-1)\cdots(n_{j_h}-1)} \cdot \left\{\sigma_0^2\right\}^{(n_0-1)n_1 n_2 \cdots n_r}, \tag{3.11}$$

which is equal to (3.9).

3.2. The inverse of the variance matrix.

At first we observe the recurrence relation of $A_{(1)}B_{(J_h)}^{(V_e)}$.

LEMMA 3.1.

$$A_{(1)}B_{(J_h)}^{(V_e)} = \frac{1}{n_{v_e}} \left[A_{(1)}B_{(J_h)}^{(V_e-1)} - A_{(1)}B_{(J_h v_e)}^{(V_e-1)} \right]. \tag{3.12}$$

This lemma enables us to express $A_{(1)}B_{(J_h)}^{(V_e)}$ in terms of $A_{(1)}B_{(J_h T_q)}$, which is given by

LEMMA 3.2.

$$A_{(1)}B_{(J_h)}^{(V_e)} = \frac{1}{n_{v_1} n_{v_2} \cdots n_{v_e}} \sum_{q=0}^{e} \sum_{T_q \subset V_e} (-1)^q A_{(1)} B_{(J_h, T_q)}. \tag{3.13}$$

The above two lemata are easily proved in the similar way to Section 4 of [2].
Now let us derive the inversion of the variance matrix.

THEOREM 3.2. *The inverse of the variance matrix* (3.2) *is given by*

$$X_1 E_{n_0} \otimes \prod_{\zeta=1}^{r} \otimes \left(E_{n_\zeta}^{1-\delta_1^\zeta} \times I_{n_\zeta}^{\delta_1^\zeta} \right) + \sum_{h=1}^{r-1} \sum_{J_h \subset R-1} X_{1J_h} E_{n_0} \otimes \prod_{\zeta=1}^{r} \otimes \left(E_{n_\zeta}^{1-\delta_{1J_h}^\zeta} \times I_{n_\zeta}^{\delta_{1J_h}^\zeta} \right) + X_0 I_{n_0} \otimes I_{n_1} \otimes \cdots \otimes I_{n_r}, \tag{3.14}$$

where

$$X_0 = \frac{1}{\sigma_0^2}, \tag{3.15}$$

$$X_R = \frac{1}{n_0} \left[\frac{1}{C_{(1)} D_{(R-1)}} - \frac{1}{\sigma_0^2} \right], \tag{3.16}$$

$$X_1 = \frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_\zeta^1}} \left[\sum_{\beta=0}^{r-1} \sum_{N\beta \subset R-1} \frac{(-1)^\beta}{C_{(1)} D_{(N\beta)}} \right], \tag{3.17}$$

$$X_{1J_h} = \frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta}} \left[\sum_{\beta=0}^{r-1-h} \sum_{N\beta \subset R-1-J_h} \frac{(-1)^\beta}{C_{(1)} D_{(J_h N\beta)}} \right], \qquad (J_h \subset R-1;\ h=1, 2, \cdots, r-1). \tag{3.18}$$

PROOF. Anticipating the inverse to be the matrix of the form

$$X_0 E_{n_0} \otimes E_{n_1} \otimes \cdots \otimes E_{n_r} + X_1 E_{n_0} \otimes \prod_{\zeta=1}^{r} \otimes \left(E_{n_\zeta}^{1-\delta_1^\zeta} \times I_{n_\zeta}^{\delta_1^\zeta} \right) \tag{3.19}$$

$$+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1} X_{1J_h}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{1J_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{1J_h}^{\zeta}}\right)$$

$$+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1} X_{J_h}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{J_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{J_h}^{\zeta}}\right)+X_0I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r},$$

let us seek for the condition that (3.19) is actually the inverse of (3.2). The product of the variance matrix (3.2) and the matrix (3.19) is

$$(3.20)\quad E_{n_0}\otimes E_{n_1}\otimes\cdots\otimes E_{n_r}\Bigg[X_G\Bigg\{\sum_{\beta=0}^{r-1}\sum_{N\beta\subset R-1}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1N\beta}^{\zeta}}+\sigma_0^2\Bigg\}$$

$$+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}X_{J_h}\Bigg\{\sum_{\beta=0}^{r-1-h}\sum_{N\beta\subset R-1-J_h}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1N\beta J_h}^{\zeta}}\Bigg\}\Bigg]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_1^{\zeta}}\times I_{n_\zeta}^{\delta_1^{\zeta}}\right)\Bigg[X_1\Bigg\{\sum_{\beta=0}^{r-1}\sum_{N\beta\supset R-1}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1N\beta}^{\zeta}}+\sigma_0^2\Bigg\}$$

$$+\sum_{e=1}^{r-1}\sum_{V_e\subset R-1}X_{1V_e}\Bigg\{\sum_{\beta}^{r-1-e}\sum_{N\beta\subset R-1-V_e}\sigma_{1N\beta}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1N\beta V_e}^{\zeta}}\Bigg\}+X_0\sigma_1^2\Bigg]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_R^{\zeta}}\times I_{n_\zeta}^{\delta_R^{\zeta}}\right)\Big[X_R(n_0\sigma_R^2+\sigma_0^2)+X_0\sigma_R^2\Big]$$

$$+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1}E_{n_0}\otimes I_{n_1}\otimes\prod_{\zeta=2}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{J_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{J_h}^{\zeta}}\right)\Bigg[X_{1J_h}\Bigg\{\sum_{\beta=h}^{r-1}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-1}}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1N\beta}^{\zeta}}+\sigma_0^2\Bigg\}$$

$$+\sum_{e=1}^{r-1-h}\sum_{V_e\subset R-1-J_h}X_{1J_hV_e}\Bigg\{\sum_{\beta=h}^{r-1-e}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-1-V_e}}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1N\beta V_e}^{\zeta}}\Bigg\}+X_0\sigma_{1J_h}^2\Bigg]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{R-1}^{\zeta}}\times I_{n_\zeta}^{\delta_{R-1}^{\zeta}}\right)\Big[X_{R-1}\{n_0\sigma_R^2+\sigma_0^2\}\Big]$$

$$+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{J_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{J_h}^{\zeta}}\right)\Bigg[X_{J_h}\Bigg\{\sum_{\beta=h}^{r-1}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-1}}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1N\beta}^{\zeta}}+\sigma_0^2\Bigg\}$$

$$+\sum_{e=1}^{r-1-h}\sum_{V_e\subset R-1-J_h}X_{J_hV_e}\Bigg\{\sum_{\beta=h}^{r-1-e}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-1-V_e}}\sigma_{1N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1N\beta V_e}^{\zeta}}\Bigg\}\Bigg]$$

$$+I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r}X_0\sigma_0^2$$

$$=E_{n_0}\otimes E_{n_1}\otimes\cdots\otimes E_{n_r}\Bigg[X_G\{C_{(1)}D\}+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}X_{J_h}\{A_{(1)}B^{(J_h)}\}\Bigg]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_1^{\zeta}}\times I_{n_\zeta}^{\delta_1^{\zeta}}\right)\Bigg[X_1\{C_{(1)}D\}+\sum_{e=1}^{r-1}\sum_{V_e\subset R-1}X_{1V_e}\{A_{(1)}B^{(V_e)}\}+X_0\sigma_1^2\Bigg]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_R^{\zeta}}\times I_{n_\zeta}^{\delta_R^{\zeta}}\right)\Big[X_R\{C_{(1)}D_{(R-1)}\}+X_0\sigma_R^2\Big]$$

$$+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1} E_{n_0}\otimes I_{n_1}\otimes\prod_{\zeta=2}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{J_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{J_h}^{\zeta}}\right)\Big[X_{1J_h}\{C_{(1)}D_{(J_h)}\}$$

$$+\sum_{e=1}^{r-1-h}\sum_{V_e\subset R-1-J_h} X_{1J_hV_e}\{A_{(1)}B_{(J_h)}^{(V_e)}\}+X_0\sigma_{1J_h}^2\Big]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{R-1}^{\zeta}}\times I_{n_\zeta}^{\delta_{R-1}^{\zeta}}\right)\Big[X_{R-1}\{C_{(1)}D_{(R-1)}\}\Big]$$

$$+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1} E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{J_h}^{\zeta}}\times I_{n_\zeta}^{\delta_{J_h}^{\zeta}}\right)\Big[X_{J_h}\{C_{(1)}D_{(J_h)}\}+\sum_{e=1}^{r-1-h}\sum_{V_e\subset R-1-J_h} X_{J_hV_e}\{A_{(1)}B_{(J_h)}^{(V_e)}\}\Big]$$

$$+I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r}X_0\sigma_0^2.$$

Thus the condition we are seeking for is expressed by the following equations,

(3.21) $$X_0\sigma_0^2=1,$$

(3.22) $$X_R\{C_{(1)}D_{(R-1)}\}=-X_0\sigma_R^2,$$

(3.23) $$X_{1J_h}\{C_{(1)}D_{(J_h)}\}=-\sum_{e=1}^{r-1-h}\sum_{V_e\subset R-1-J_h} X_{1J_hV_e}\{A_{(1)}B_{(J_h)}^{(V_e)}\}-X_0\sigma_{1J_h}^2,$$

$$(J_h\subset R-1;\ h=1,2,\cdots,r-2).$$

(3.24) $$X_1\{C_{(1)}D\}=-\sum_{e=1}^{r-1}\sum_{V_e\subset R-1} X_{1V_e}\{A_{(1)}B^{(V_e)}\}-X_0\sigma_1^2,$$

(3.25) $$X_{R-1}\{C_{(1)}D_{(R-1)}\}=0,$$

(3.26) $$X_{J_h}\{C_{(1)}D_{(J_h)}\}=-\sum_{e=1}^{r-1-h}\sum_{V_e\subset R-1-J_h} X_{J_hV_e}\{A_{(1)}B_{(J_h)}^{(V_e)}\},$$

$$(J_h\subset R-1;\ h=1,2,\cdots,r-2),$$

(3.27) $$X_G\{C_{(1)}D\}=-\sum_{h=1}^{r-1}\sum_{J_h\subset R-1} X_{J_h}\{A_{(1)}B^{(J_h)}\}.$$

The solutions of the last three systems of the equations (3.25), (3.26) and (3.27) are given as follows, it is obvious, by solving successively,

(3.28) $$X_{J_h}=0, \qquad (J_h\subset R-1;\ h=1,2,\cdots,r-1),$$

$$X_G=0.$$

Since, as is easily seen, to solve the remaining four systems of the equations, (3.21), . (3.24) corresponds to solving (4.27), (4.28), (4.29) in [2], the solutions of (3.21), . (3.24) are given by (3.15), (3.16), (3.17) and (3.18).

Thus the proof of this theorem is completed.

3.3. The joint density function.

The joint density function in Type I is given in the following.

THEOREM 3.3. *The joint density function of all observations in our case is given by*

$$f(X)=(2\pi)^{-n_0n_1\cdots n_r/2}\prod_{h=0}^{r-1}\prod_{J_h\subset R-1}\{C_{(1)}D_{(J_h)}\}^{-n_1(n_{j_1}-1)\cdots(n_{j_h}-1)/2}\{\sigma_0^2\}^{-(n_0-1)n_1\cdots n_r/2} \tag{3.29}$$

$$\cdot\ exp\left[-\frac{1}{2}\left\{\frac{S'_{(1)}}{C_{(1)}D}+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\frac{S'_{(1J_h)}}{C_{(1)}D_{(J_h)}}+\frac{S_0}{\sigma_0^2}\right\}\right],$$

where

$$\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1L_\beta}^{\zeta}}}\sum_{\substack{t_{i_1},\cdots,t_{i_{r-1-\beta}}\\ I_{r-1-\beta}\subset R-1-L_\beta}}\sum_{t_0}x_{t_0t_1\cdots t_r},\qquad (L_\beta\subset R-1;\ \beta=0,1,\cdots,r-1), \tag{3.30}$$

$$S'_{(1)}=\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_1^\zeta}\sum_{t_1=1}^{n_1}(\bar{X}_{t_1}-\mu)^2, \tag{3.31}$$

$$S'_{(1J_h)}=\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1J_h}^{\zeta}}\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\left\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}X_{t_1t_{l_1}\cdots t_{l\beta}}-\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})\right\}^2, \tag{3.32}$$

$$(J_h\subset R-1;\ h=1,\cdots,r-1),$$

$$S_0=\sum_{t_0t_1\cdots t_r}(x_{t_0t_1\cdots t_r}-\bar{X}_{t_0t_1\cdots t_r})^2, \tag{3.33}$$

where in (3.30) $\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}=\bar{X}_{t_1}$ *when* $\beta=0$.

PROOF. As it is obvious that the type of the density function is the normal distribution, the constant factor in (3.29) is easily derived from Theorem 3.2. Before evaluating the quadratic form of $x_{t_0t_1\cdots t_r}$, let us introduce new variables defined by

$$u_{t_0t_1\cdots t_r}=x_{t_0t_1\cdots t_r}-\mu-\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h}), \tag{3.34}$$

$$U_{t_1t_{l_1}\cdots t_{l\beta}}=\sum_{\substack{t_{d_1},\cdots,t_{d_{r-1-\beta}}\\ D_{r-1-\beta}\subset R-1-L_\beta}}\sum_{t_0}u_{t_0t_1\cdots t_r},\qquad (L_\beta\subset R-1;\ \beta=0,\cdots,r-1), \tag{3.35}$$

$$\bar{U}_{t_1t_{l_1}\cdots t_{l\beta}}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1L_\beta}^{\zeta}}}U_{t_1t_{l_1}\cdots t_{l\beta}},\qquad (L_\beta\subset R-1;\ \beta=0,\cdots,r-1), \tag{3.36}$$

where when $\beta=0$ we shall use the convention for $U_{t_1t_{l_1}\cdots t_{l\beta}}$ and $\bar{U}_{t_1t_{l_1}\cdots t_{l\beta}}$ same to that stated in this theorem. Furthermore, let us remark that (3.36) may be expressed in terms of $\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}$, μ and $\alpha(v_1,v_2,\cdots,v_p;\ t_{v_1},t_{v_2},\cdots,t_{v_p})$, by the assumptions (2.2), as follows,

$$\bar{U}_{t_1t_{l_1}\cdots t_{l\beta}}=\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}-\mu-\sum_{p=1}^{\beta}\sum_{V_p\subset L_\beta}\alpha(v_1,\cdots,v_p;\ t_{v_1},\cdots,t_{v_p}), \tag{3.37}$$

$$\bar{U}_{t_1}=\bar{X}_{t_1}-\mu. \tag{3.38}$$

Using the inverse matrix derived in Theorem 3.2 we have the term of the quadratic form in the joint density function that

$$(3.39)\quad S=X_1\left\{\sum_{t_1}\left(\sum_{t_0,t_2,\cdots,t_r}u_{t_0t_1\cdots t_r}\right)^2\right\}+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1}X_{1J_h}\left\{\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\left(\sum_{\substack{t_0,t_{d_1},\cdots,t_{d_{r-h-1}}\\ D_{r-h-1}\subset R-}}u_{t_0t_1\cdots t_r}\right)^2\right\}$$

$$+X_R\sum_{t_1,\cdots,t_r}\left(\sum_{t_0}u_{t_0t_1\cdots t_r}\right)^2+X_0\sum_{t_0,t_1,\cdots,t_r}u^2_{t_0t_1\cdots t_r}$$

$$=X_1\sum_{t_1}U^2_{t_1}+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1}X_{1J_h}\left\{\sum_{t_1,t_{j_1},\cdots,t_{j_h}}U^2_{t_1t_{j_1}\cdots t_{j_h}}\right\}$$

$$+X_R\sum_{t_1,\cdots,t_r}U^2_{t_1\cdots t_r}+X_0\sum_{t_0,t_1,\cdots,t_r}u^2_{t_0t_1\cdots t_r}$$

$$=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_1^\zeta}}\left[\sum_{\beta=0}^{r-1}\sum_{N\beta\subset R-1}\frac{(-1)^\beta}{C_{(1)}D_{(N\beta)}}\right]\left(\sum_{t_1}U^2_{t_1}\right)$$

$$+\sum_{h=1}^{r-2}\sum_{J_h\subset R-1}\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1J_h}^\zeta}}\left[\sum_{\beta=0}^{r-1-h}\sum_{N\beta\subset R-1-J_h}\frac{(-1)^\beta}{C_{(1)}D_{(J_hN\beta)}}\right]\left(\sum_{t_1,t_{j_1},\cdots,t_{j_h}}U^2_{t_1t_{j_1}\cdots t_{j_h}}\right)$$

$$+\frac{1}{n_0}\left[\frac{1}{C_{(1)}D_{(R-1)}}-\frac{1}{\sigma_0^2}\right]\left(\sum_{t_1,\cdots,t_r}U^2_{t_1\cdots t_r}\right)+\frac{1}{\sigma_0^2}\left(\sum_{t_0,\cdots,t_r}u^2_{t_0t_1\cdots t_r}\right).$$

The coefficients to $\frac{1}{C_{(1)}D}$, $\frac{1}{C_{(1)}D_{(L\beta)}}$ and $\frac{1}{C_{(1)}D_{(R-1)}}$ in (3.39) are given by

$$(3.40)\qquad \frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_1^\zeta}}\left(\sum_{t_1}U^2_{t_1}\right),$$

$$(3.41)\qquad \sum_{h=0}^{\beta}\sum_{J_h\subset L\beta}\frac{(-1)^{\beta-h}}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1J_h}^\zeta}}\left(\sum_{t_1,t_{j_1},\cdots,t_{j_h}}U^2_{t_1t_{j_1}\cdots t_{j_h}}\right),$$

and

$$(3.42)\qquad \sum_{h=0}^{r-1}\sum_{J_h\subset R-1}\frac{(-)^{r-1-h}}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1J_h}^\zeta}}\left(\sum_{t_1,t_{j_1},\cdots,t_{j_h}}U^2_{t_1t_{j_1}\cdots t_{j_h}}\right),$$

respectively.

Therefore, by Lemma 4.4 in [2], (3.39) is equal to

$$(3.43)\quad \sum_{\beta=1}^{r-1}\sum_{L\beta\subset R-1}\sum_{h=0}^{\beta}\sum_{J_h\subset L\beta}\frac{(-1)^{\beta-h}}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1J_h}^\zeta}}\left(\sum_{t_1,t_{j_1},\cdots,t_{j_h}}U^2_{t_1t_{j_1}\cdots t_{j_h}}\right)\frac{1}{C_{(1)}D_{(L\beta)}}$$

$$+\left[\sum_{t_0,t_1,\cdots,t_r}u^2_{t_0t_1\cdots t_r}-\frac{1}{n_0}\sum_{t_1\cdots t_r}U^2_{t_1\cdots t_r}\right]\frac{1}{\sigma_0^2}+\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_1^\zeta}}\left(\sum_{t_1}U^2_{t_1}\right)\frac{1}{C_{(1)}D}$$

$$=\sum_{\beta=1}^{r-1}\sum_{L\beta\subset R-1}\sum_{h=0}^{\beta}\sum_{J_h\subset L\beta}(-1)^{\beta-h}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{1J_h}^\zeta}\left(\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\bar{U}_{t_1t_{j_1}\cdots t_{j_h}}\right)\frac{1}{C_{(1)}D_{(L\beta)}}$$

$$+\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}\Big(\sum_{t_1}\bar{U}_{t_1}^2\Big)\frac{1}{C_{(1)}D}+\Big[\sum_{t_0,t_1,\cdots,t_r} u_{t_0t_1\cdots t_r}^2-n_0\sum_{t_1}\bar{U}_{t_1}^2\Big]\frac{1}{\sigma_0^2}$$

$$=\sum_{\beta=1}^{r-1}\sum_{L\beta\subset R-1}\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1L\beta}^\zeta}\Big[\sum_{t_1,t_{l_1},\cdots,t_{l\beta}}\sum_{h=0}^{\beta}\sum_{J_h\subset L\beta}(-1)^{\beta-h}\bar{U}_{t_1t_{j_1}\cdots t_{j_h}}^2\Big]\frac{1}{C_{(1)}D_{(L\beta)}}$$

$$+\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}\Big(\sum_{t_1}\bar{U}_{t_1}^2\Big)\frac{1}{C_{(1)}D}+\Big[\sum_{t_0,t_1,\cdots,t_r} u_{t_0t_1\cdots t_r}^2-n_0\sum_{t_1\cdots t_r}\bar{U}_{t_1\cdots t_r}^2\Big]\frac{1}{\sigma_0^2}$$

$$=\sum_{\beta=1}^{r-1}\sum_{L\beta\subset R-1}\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1L\beta}^\zeta}\Big[\sum_{t_1,t_{l_1},\cdots,t_{l\beta}}\Big\{\sum_{h=0}^{\beta}\sum_{J_h\subset L\beta}(-1)^{\beta-h}\bar{U}_{t_1t_{j_1}\cdots t_{j_h}}\Big\}^2\Big]\frac{1}{C_{(1)}D_{(L\beta)}}$$

$$+\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}\Big(\sum_{t_1}\bar{U}_{t_1}^2\Big)\frac{1}{C_{(1)}D}+\sum_{t_0,t_1,\cdots,t_r}(u_{t_0t_1\cdots t_r}-\bar{U}_{t_1\cdots t_r})^2\frac{1}{\sigma_0^2}$$

$$=\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta}\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\Big\{\sum_{\beta=0}^{h}\sum_{L\beta\subset J_h}(-1)^{h-\beta}\bar{U}_{t_1t_{l_1}\cdots t_{l\beta}}\Big\}^2\frac{1}{C_{(1)}D_{(J_h)}}$$

$$+\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}\Big(\sum_{t_1}\bar{U}_{t_1}^2\Big)\frac{1}{C_{(1)}D}+\sum_{t_0,t_1,\cdots,t_r}(u_{t_0t_1\cdots t_r}-\bar{U}_{t_1\cdots t_r})^2\frac{1}{\sigma_0^2}.$$

After inserting (3.37) and (3.38) in the above formula, (3.43) is given by

(3.44)

$$\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta}\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\Big[\sum_{\beta=0}^{h}\sum_{L\beta\subset J_h}(-1)^{h-\beta}\Big\{\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}-\mu-\sum_{p=1}^{\beta}\sum_{V_p\subset L\beta}\alpha(v_1,\cdots,v_p;t_{v_1},\cdots,t_{v_p})\Big\}\Big]^2$$

$$\cdot\frac{1}{C_{(1)}D_{(J_h)}}+\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta}\Big\{\sum_{t_1}(\bar{X}_{t_1}-\mu)^2\Big\}\frac{1}{C_{(1)}D}$$

$$+\sum_{t_0,\cdots,t_r}(x_{t_0t_1\cdots t_r}-\bar{X}_{t_1\cdots t_r})^2\frac{1}{\sigma_0^2}.$$

As the formula which is squared in the first term is simplified as follows,

$$(3.45)\quad \sum_{\beta=0}^{h}\sum_{L\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}-\sum_{\beta=0}^{h}\sum_{L\beta\subset J_h}(-1)^{h-\beta}\sum_{p=1}^{\beta}\sum_{V_p\subset L\beta}\alpha(v_1,v_2,\cdots,v_p;\ t_{v_1},t_{v_2},\cdots,t_{v_p})$$

$$=\sum_{\beta=0}^{h}\sum_{L\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}-\sum_{k=1}^{h-1}\sum_{V_k\subset J_h}\{{}_{h-k}C_0(-1)^{h-k}+{}_{h-k}C_1(-1)^{h-k-1}+\cdots+{}_{h-k}C_{h-k}(-1)^0\}$$

$$\cdot\ \alpha(v_1,\cdots,v_k;\ t_{v_1},\cdots,t_{v_k})-\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})$$

$$=\sum_{\beta=0}^{h}\sum_{L\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}-\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h}),$$

finally (3.43) is equal to

(3.46)

$$\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta}\sum_{t_1,t_{j_1},\cdots,t_{j_h}}\Big\{\sum_{\beta=0}^{h}\sum_{L\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_1t_{l_1}\cdots t_{l\beta}}-\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})\Big\}^2\frac{1}{C_{(1)}D_{(J_h)}}$$

$$+\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta} \sum_{t_1} (\bar{X}_{t_1}-\mu)^2 \frac{1}{C_{(1)}D} + \sum_{t_0, t_1, \cdots, t_r} (x_{t_0 t_1 \cdots t_r} - \bar{X}_{t_1 \cdots t_r})^2 \frac{1}{\sigma_0^2},$$

which completes the theorem.

3.4. Estimation.

In order to show the sufficient statistic for the distribution whose joint density function is given by Theorem 3.3 and to derive the density function of the sufficient statistic, we need to modificate (3.29) further.

Now we have

LEMMA 3.3. *The joint density function of all observations,* (3.29), *is equal to*

$$f(X) = K\varphi(\sigma^2, \alpha, \mu)$$

$$\cdot \exp\Big[Z^{(1)} \frac{\mu}{C_{(1)}D} + \sum_{h=1}^{r-1} \sum_{J_h \subset R-1} \sum_{t_{j_1}=1}^{n_{j_1}-1} \sum_{t_{j_2}=1}^{n_{j_2}-1} \cdots \sum_{t_{j_h}=1}^{n_{j_h}-1} Z^{(2)}_{(J_h; t_{j_1}, \cdots, t_{j_h})} \frac{\alpha(j_1, \cdots, j_h; t_{j_1}, \cdots, t_{j_h})}{C_{(1)} D_{(J_h)}} \tag{3.47}$$

$$- \frac{1}{2} Z^{(3)} \frac{1}{C_{(1)}D} - \frac{1}{2} \sum_{h=1}^{r-1} \sum_{J_h \subset R-1} Z^{(4)}_{(J_h)} \frac{1}{C_{(1)}D_{(J_h)}} - \frac{S_0}{2\sigma_0^2}\Big],$$

where

$$Z^{(1)} = \prod_{i=0}^{r} n_i \bar{X}, \tag{3.48}$$

$$Z^{(2)}_{(J_h; t_{j_1}, \cdots, t_{j_h})} = \prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{J_h}^\zeta} \Big(\bar{X}_{t_{j_1} \cdots t_{j_h}} + \sum_{\gamma=1}^{h} \sum_{D_{h-\gamma} \subset J_h} (-1)^\gamma \bar{X}_{t_{d_1} \cdots t_{d_{h-\gamma}}\, n_{d_{h-\gamma+1}} \cdots n_{d_h}}\Big), \tag{3.49}$$

$$Z^{(3)} = \prod_{\zeta=0}^{r} n_\zeta^{1-\delta_1^\zeta} \Big(\sum_{t_1} \bar{X}_{t_1}^2\Big), \tag{3.50}$$

$$Z^{(4)}_{(J_h)} = \prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{1J_h}^\zeta} \sum_{t_1, t_{j_1}, \cdots, t_{j_h}} \Big(\sum_{\beta=0}^{h} \sum_{L_\beta \subset J_h} (-1)^{h-\beta} \bar{X}_{t_1 t_{l_1} \cdots t_{l_\beta}}\Big)^2. \tag{3.51}$$

PROOF. It should be enough to work out on (3.32), which is equal to except for the constant term, in virtue of (2.2),

$$\sum_{t_1, t_{j_1}, \cdots, t_{j_h}} \Big\{\sum_{\beta=0}^{h} \sum_{L_\beta \subset J_h} (-1)^{h-\beta} \bar{X}_{t_1 t_{l_1} \cdots t_{l_\beta}}\Big\}^2 \tag{3.52}$$

$$-2 \sum_{t_1, t_{j_1}, \cdots, t_{j_h}} \Big\{\sum_{\beta=0}^{h} \sum_{L_\beta \subset J_h} (-1)^{h-\beta} \bar{X}_{t_1 t_{l_1} \cdots t_{l_\beta}}\Big\} \alpha(j_1, \cdots, j_h;\ t_{j_1}, \cdots, t_{j_h}) + g(\alpha)$$

$$= \sum_{t_1, t_{j_1}, \cdots, t_{j_h}} \Big\{\sum_{\beta=0}^{h} \sum_{L_\beta \subset J_h} (-1)^{h-\beta} \bar{X}_{t_1 t_{l_1} \cdots t_{l_\beta}}\Big\}^2$$

$$-2n_1 \sum_{t_{j_1}, \cdots, t_{j_h}} \bar{X}_{t_{j_1} \cdots t_{j_h}} \alpha(j_1, \cdots, j_h;\ t_{j_1}, \cdots, t_{j_h}) + g(\alpha).$$

On the other hand we have that

$$\sum_{t_{j_1}, \cdots, t_{j_k}} \bar{X}_{t_{j_1} \cdots t_{j_k}} \alpha(j_1, \cdots, j_k;\ t_{j_1}, \cdots, t_{j_k}) \tag{3.53}$$

$$=\sum_{t_{j_1}=1}^{n_{j_1}-1}\sum_{t_{j_2}=1}^{n_{j_2}-1}\cdots\sum_{t_{j_k}=1}^{n_{j_k}-1}\left\{X_{t_{j_1}\cdots t_{j_k}}+\sum_{\gamma=1}^{k}\sum_{D_{k-\gamma}\subset J_k}(-1)^{\gamma}\bar{X}_{t_{d_1}\cdots t_{d_{k-\gamma}}\,n_{d_{k-\gamma+1}}\cdots n_{d_k}}\right\}$$

$$\cdot\,\alpha(j_1,\cdots,j_k;\ t_{j_1},\cdots,t_{j_k}),\qquad (J_k\subset R-1;\ k=1,2,\cdots,r-1),$$

for, when $k=1$

$$\sum_{t_{j_1}}\bar{X}_{t_{j_1}}\alpha(j_1;t_{j_1})=\sum_{t_{j_1}=1}^{n_{j_1}-1}X_{t_{j_1}}\alpha(j_1;t_{j_1})+\bar{X}_{n_{j_1}}\alpha(j_1;n_{j_1}) \tag{3.54}$$

$$=\sum_{t_{j_1}=1}^{n_{j_1}-1}\bar{X}_{t_{j_1}}\alpha(j_1;t_{j_1})+\bar{X}_{n_{j_1}}\left(-\sum_{t_{j_1}=1}^{n_{j_1}-1}\alpha(j_1;t_{j_1})\right)$$

$$=\sum_{t_{j_1}=1}^{n_{j_1}-1}(\bar{X}_{t_{j_1}}-\bar{X}_{n_{j_1}})\ \alpha(j_1;t_{j_1}),$$

and when we assume that (3.51) holds true for $k=c$

$$\sum_{t_{j_1},\cdots,t_{j_{c+1}}}\bar{X}_{t_{j_1}\cdots t_{j_{c+1}}}\alpha(j_1\cdots j_{c+1};\ t_{j_1}\cdots t_{j_{c+1}}) \tag{3.55}$$

$$=\sum_{t_{j_1}}\cdots\sum_{t_{j_c}}\left\{\sum_{t_{j_{c+1}}=1}^{n_{j_{c+1}}-1}\bar{X}_{t_{j_1}\cdots t_{j_{c+1}}}\alpha(j_1,\cdots,j_{c+1};\ t_{j_1},\cdots,t_{j_{c+1}})\right.$$

$$\left.+\bar{X}_{t_{j_1}\cdots t_{j_c}n_{j_{c+1}}}\alpha(j_1,\cdots,j_{c+1};\ t_{j_1},\cdots,t_{j_c},n_{j_{c+1}})\right\}$$

$$=\sum_{t_{j_1},\cdots,t_{j_c}}\left\{\sum_{t_{j_{c+1}}=1}^{n_{j_{c+1}}-1}(\bar{X}_{t_{j_1}\cdots t_{j_{c+1}}}-\bar{X}_{t_{j_1}\cdots t_{j_c}n_{j_{c+1}}})\alpha(j_1,\cdots,j_{c+1};\ t_{j_1},\cdots,t_{j_{c+1}})\right\}$$

$$=\sum_{t_{j_1}=1}^{n_{j_1}-1}\cdots\sum_{t_{j_{c+1}}=1}^{n_{j_{c+1}}-1}\left\{\bar{X}_{t_{j_1}\cdots t_{j_{c+1}}}+\sum_{\gamma=1}^{c}\sum_{D_{c-\gamma}\subset J_c}(-1)^{\gamma}\bar{X}_{t_{d_1}\cdots t_{d_{c-\gamma}}\,n_{d_{c-\gamma+1}}\cdots n_{d_c}t_{j_{c+1}}}\right\}$$

$$\cdot\,\alpha(j_1,\cdots,j_{c+1};\ t_{j_1},\cdots,t_{j_{c+1}})$$

$$-\sum_{t_{j_1}=1}^{n_{j_1}-1}\cdots\sum_{t_{j_c}=1}^{n_{j_c}-1}\left\{\bar{X}_{t_{j_1}\cdots t_{j_c}n_{j_{c+1}}}+\sum_{\gamma=1}^{c}\sum_{D_{c-\gamma}\subset J_c}(-1)^{\gamma}\bar{X}_{t_{d_1}\cdots t_{d_{c-\gamma}}\,n_{d_{c-\gamma+1}}\cdots n_{d_c}n_{j_{c+1}}}\right\}$$

$$\cdot\,\alpha(j_1,\cdots,j_{c+1};\ t_{j_1},\cdots,t_{j_{c+1}})$$

$$=\sum_{t_{j_1}=1}^{n_{j_1}-1}\cdots\sum_{t_{j_{c+1}}=1}^{n_{j_{c+1}}-1}\left\{\bar{X}_{t_{j_1}\cdots t_{j_{c+1}}}+\sum_{\gamma=1}^{c+1}\sum_{D_{c+1-\gamma}\subset J_{c+1}}(-1)^{\gamma}\bar{X}_{t_{d_1}\cdots t_{d_{c+1-\gamma}}\,n_{d_{c+2-\gamma}}\cdots n_{j_{c+1}}}\right\}$$

$$\cdot\,\alpha(j_1,\cdots,j_{c+1};\ t_{j_1},\cdots,t_{j_{c+1}}),$$

therefore (3.52) should be given by

$$\sum_{t_{j_1}}\cdots\sum_{t_{j_h}}\left\{\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_{l_1}\cdots t_{l_\beta}}\right\}^2$$

$$-2n_1\sum_{t_{j_1}=1}^{n_{j_1}-1}\cdots\sum_{t_{j_h}=1}^{n_{j_h}-1}\left\{\bar{X}_{t_{j_1}\cdots t_{j_h}}+\sum_{\gamma=1}^{h}\sum_{D_{h-\gamma}\subset J_h}(-)^{\gamma}\bar{X}_{t_{d_1}\cdots t_{d_{h-\gamma}}\,n_{d_{h-\gamma+1}}\cdots n_{d_h}}\right\}$$

$$\cdot\,\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})\ +g(\alpha).$$

This completes the proof of Lemma 3.3.

For the development of the arguments we need to define a number of notations and consider about the completeness of the distributions of the sufficient statistics.

Let X be a random variable, R^X be a sample space of X, P_θ^X be the probability distribution of X defined over R^X, which is indexed by a subscript θ taking its value in an abstract space Ω, and $\mathfrak{P}^X=\{P_\theta^X \mid \theta\in\Omega\}$ be a family of the probability distributions of X. And let $U=u(X)$ be the sufficient statistic for $\mathfrak{P}^X$, R^U be a space of the sufficient statistic U and $\mathfrak{P}^U=\{P_\theta^U \mid \theta\in\Omega\}$ be a family of the probability distributions of U defined over R_U.

We remark the well-known result, (see [6], [7]), that if there exists a sufficient statistic U for $\mathfrak{P}^X$ such that $\mathfrak{P}^U$ is complete then a statistic is a minimum variance estimate of its expected value if and only if it is the function of U (a. e. $\mathfrak{P}^X$), and yet it is unique for U.

As for completeness, we have the following theorem:
If $X=(X_1, X_2, \cdots, X_n)$, $\theta=(\theta_1, \theta_2, \cdots, \theta_k)$ if Ω contains a non-degenerate k-dimensional interval, and if P_θ^X has a density of the form

$$dP_\theta^X \Big/ d\mu^X = C(\theta)exp\left[\sum_{i=1}^{k}\theta_i u_i(X)\right] \tag{3.56}$$

with respect to a measure μ^X on the class $\mathfrak{F}^X$ of Borel sets in the n-dimensional Euclidean space W^X, then $U=(u_1(X), u_2(X), \cdots, u_k(X))$ is a sufficient statistic for $\mathfrak{P}^X$ with a probability density

$$dP_\theta^U \Big/ d\nu^U = C'(\theta)h(u)exp\left[\sum_{i=1}^{k}\theta_i u_i\right] \tag{3.57}$$

with respect to a measure ν^U on R^U, and the family $\mathfrak{P}_U$ is strongly complete; (see [7]).

In our mixed model of Type I, the random variable is $X=(X_{11\cdots1}, \cdots, X_{n_011\cdots1}; X_{121\cdots1}, \cdots, X_{n_021\cdots1}; \cdots\cdots; X_{1n_1\cdots n_r}, \cdots, X_{n_0n_1\cdots n_r})$, the sample space R^X is a $n_0n_1\cdots n_r$-dimensional Euclidean space, and the family $\mathfrak{P}^X$ is specified by the parameter $\theta=(\mu, \alpha(j_1, \cdots, j_h; t_{j_1}, \cdots, t_{j_h}), \sigma_1^2, \sigma_{1J_h}^2, \sigma_0^2; J_h\subset R-1, t_{j_c}=1, \cdots, n_{j_c}-1, c=1, \cdots, h, h=1, \cdots, r-1)$ whose space is of $(2^{r-1}+1+n_1n_2\cdots n_r)$-dimension, where $-\infty<\mu<\infty$, $-\infty<\alpha(j_1, \cdots, j_h; t_{j_1}, \cdots, t_{j_h})<\infty$, $0<\sigma_0^2<\infty$, $0<\sigma_1^2<\infty$, $0<\sigma_{1J_h}^2<\infty$.

As we have seen in Lemma 3.3, the probability density of X is given by (3.47). In order to derive the sufficient statistic for $\mathfrak{P}^X$, and to show the completeness of $\mathfrak{P}^U$, we consider the transformations of the original parameters such that

$$\tau^{(1)}=\frac{\mu}{C_{(1)}D}, \tag{3.58}$$

$$\tau^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})}=\frac{\alpha(j_1, \cdots, j_h; t_{j_1}, \cdots, t_{j_h})}{C_{(1)}D_{(J_h)}}, \qquad \begin{pmatrix} J_h\subset R-1; \ t_{j_c}=1, \cdots, n_{j_c}-1; \\ c=1, \cdots, h; \ h=1, \cdots, r-1 \end{pmatrix} \tag{3.59}$$

$$\tau^{(3)}=-\frac{1}{2C_{(1)}D}, \tag{3.60}$$

$$(3.61)\qquad \tau^{(4)}_{(J_h)}=-\frac{1}{2C_{(1)}D_{(J_h)}}, \qquad (J_h\subset R-1;\ h=1,\cdots,r-1),$$

$$(3.62)\qquad \tau^{(5)}=-\frac{1}{2\sigma_0^2}.$$

After observing the independency of the class of parsametric functions $\{C_{(1)}D_{(J_h)};\ J_h\subset R-1,\ h=1,\cdots,r-1\}$, it is noted that the transformation (3.58), ⋯, (3.62) from θ to $\tau=(\tau^{(1)},\ \tau^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})},\ \tau^{(3)},\ \tau^{(4)}_{(J_h)},\ \tau^{(5)};\ J_h\subset R-1,\ t_{j_c}=1,\ \cdots,\ n_{j_c}-1,\ c=1,\cdots,\ h,\ h=1,\cdots,\ r-1)$ is one-to-one. Therefore, considering as if the given parameter were τ instead of θ, we can say that $\mathfrak{B}^X$ is specified by τ, where $-\infty<\tau^{(1)}<\infty,\ -\infty<\tau^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})}<\infty,\ -\infty<\tau^{(5)}<\tau^{(4)}_{(M)}<\tau^{(4)}_{(N)}<\tau^{(3)}<0$ for any pair (M,N) such that $M\supset N$, $M\subset R-1$, $N\subset R-1$.

Then the probability density function of X is given in the form

$$(3.63)\qquad K\varphi_{(\tau)}exp\Big[\tau^{(1)}Z^{(1)}+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\sum_{t_{j_1}=1}^{n_{j_1}-1}\cdots\sum_{t_{j_h}=1}^{n_{j_h}-1}\tau^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})}Z^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})}$$
$$+\tau^{(3)}Z^{(3)}+\sum_{h=1}^{r-1}\sum_{J_h\subset R-1}\tau^{(4)}_{(J_h)}Z^{(4)}_{(J_h)}+\tau^{(5)}S_0\Big].$$

Therefore, from the above-reviewed results on sufficiency and completeness, the sufficient statistic for $\mathfrak{B}^X$ is $U=(Z^{(1)},\ Z^{(2)}_{(J_h;t_{j_1},\cdots,t_{j_h})},\ Z^{(3)},\ Z^{(4)}_{(J_h)},\ S_0;\ J_h\subset R-1,\ t_{j_c}=1,\cdots,\ n_{j_c}-1,\ c=1,\cdots,\ h,\ h=1,2,\cdots,\ r-1)$ and the family $\mathfrak{B}^U$ is strongly complete.

In the estimation problem of the parameters, the estimates usually adopted in the practice of statistical inferences are unbiased and based on the sufficient statistic U defined above. As we have observed its completeness, we can obtain the following:

THEOREM 3.4. *In the mixed model of Type I, the usual estimates of the parameters, such as the fixed treatment effects, the variance components of random treatment effecis, etc., are the best unbiased estimates among all unbiased estimates.*

4. The case of Type II.

4.1. The determinant of the variance matrix.

At first under the model equation (2.1) we have the expression of the variance matrix in terms of the Kronecker products as follows,

$$(4.1)\qquad V=\sum_{k=1}^{s}\sum_{I_k\subset S}\sigma^2_{I_k}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\Big(E_{n_\zeta}^{1-\delta^\zeta_{I_k}}\times I_{n_\zeta}^{\delta^\zeta_{I_k}}\Big)$$
$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\sigma^2_{I_kJ_h}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\Big(E_{n_\zeta}^{1-\delta^\zeta_{I_kJ_h}}\times I_{n_\zeta}^{\delta^\zeta_{I_kJ_h}}\Big)+\sigma_0^2I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r},$$

where $\delta^\zeta_{M_\alpha}$ is defined in (3.3).

Our next step is to introduce a number of notations given by

DEFINITION 4.1.

$$(4.2)\qquad A_{(I_k)}B_{(J_h)}\equiv\sum_{\alpha=k}^{s}\sum_{\substack{M_\alpha\supset I_k\\ M_\alpha\subset S}}\sum_{\beta=h}^{r-s}\sum_{\substack{N_\beta\supset J_h\\ N_\beta\subset R-S}}\sigma^2_{M_\alpha N_\beta}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta}},$$

$$A_{(I_k)}B \equiv \sum_{\alpha=k}^{s} \sum_{\substack{M_\alpha \supset I_k \\ M_\alpha \subset S}} \sum_{\beta=0}^{r-s} \sum_{N_\beta \subset R-S} \sigma^2_{M_\alpha N_\beta} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta}}, \tag{4.3}$$

$$AB_{(J_h)} \equiv \sum_{\alpha=1}^{s} \sum_{M_\alpha \subset S} \sum_{\beta=h}^{r-s} \sum_{\substack{N_\beta \supset J_h \\ N_\beta \subset R-S}} \sigma^2_{M_\alpha N_\beta} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta}}, \tag{4.4}$$

$$AB \equiv \sum_{\alpha=1}^{s} \sum_{M_\alpha \subset S} \sum_{\beta=0}^{r-s} \sum_{N_\beta \subset R-S} \sigma^2_{M_\alpha N_\beta} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta}}, \tag{4.5}$$

$$A^{(U_d)}_{(I_k)}B^{(V_e)}_{(J_h)} \equiv \sum_{\alpha=k}^{s-d} \sum_{\substack{M_\alpha \supset I_k \\ M_\alpha \subset S-U_d}} \sum_{\beta=h}^{r-s-e} \sum_{\substack{N_\beta \supset J_h \\ N_\beta \subset R-S-V_e}} \sigma^2_{M_\alpha N_\beta} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{M_\alpha N_\beta U_d V_e}}, \tag{4.6}$$

$$C_{(I_k)}D_{(J_h)} \equiv A_{(I_k)}B_{(J_h)} + \sigma_0^2, \tag{4.7}$$

$$C_{(I_k)}D \equiv A_{(I_k)}B + \sigma_0^2, \tag{4.8}$$

$$CD_{(J_h)} \equiv AB_{(J_h)} + \sigma_0^2, \tag{4.9}$$

$$CD \equiv AB + \sigma_0^2 \tag{4.10}$$

$$G_{(J_h)} \equiv \sum_{p=1}^{s} \sum_{L_p \subset S} (-1)^{p-1} C_{(L_p)} D_{(J_h)}, \tag{4.11}$$

$$G \equiv \sum_{p=1}^{s} \sum_{L_p \subset S} (-1)^{p} C_{(L_p)} D, \tag{4.12}$$

where $I_k \cap U_d = \phi$ and $J_h \cap V_e = \phi$.

Then we have the following relations:

$$\begin{aligned} CD_{(J_h)} &= \sum_{p=1}^{s} \sum_{L_p \subset S} (-1)^{p-1} C_{(L_p)} D_{(J_h)}, \\ CD &= \sum_{p=1}^{s} \sum_{L_p \subset S} (-1)^{p-1} C_{(L_p)} D. \end{aligned} \tag{4.13}$$

Now we can evaluate the determinant of the variance matrix (4.1).

THEOREM 4.1. *The determinant* $|V|$ *of the variance matrix* (4.1) *is given in the form*

$$\begin{aligned} & G \prod_{d=1}^{r-s} \prod_{U_d \subset R-S} \{G_{(U_d)}\}^{(n_{u_1}-1)(n_{u_2}-1)\cdots(n_{u_d}-1)} \\ & \cdot \prod_{k=1}^{s} \prod_{I_k \subset S} \prod_{h=0}^{r-s} \prod_{J_h \subset R-S} \{C_{(I_k)} D_{(J_h)}\}^{(n_{i_1}-1)\cdots(n_{i_k}-1)(n_{j_1}-1)\cdots(n_{j_h}-1)} \\ & \cdot \{\sigma_0^2\}^{(n_0-1)n_1 n_2 \cdots n_r}. \end{aligned} \tag{4.14}$$

PROOF. We shall transform the matrix (4.1) by the orthogonal matrix which is the Kronecker product of the matrixes T_{n_i} defined in Section 2, then we have

$$
\begin{aligned}
(4.15)\quad & \sum_{k=1}^{s}\sum_{I_k\subset S}\sigma_{I_k}^2\prod_{t=0}^{r}n_t^{1-\delta_{I_k}^t}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(H_{n_\zeta}^{1-\delta_{I_k}^\zeta}\times I_{n_\zeta}^{\delta_{I_k}^\zeta}\right)\\
&\quad+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\sigma_{I_kJ_h}^2\prod_{t=0}^{r}n_t^{1-\delta_{I_kJ_h}^t}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(H_{n_\zeta}^{1-\delta_{I_kJ_h}^\zeta}\times I_{n_\zeta}^{\delta_{I_kJ_h}^\zeta}\right)\\
&\quad+\sigma_0^2I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r}\\
&=\sum_{k=1}^{s}\sum_{I_k\subset S}\sigma_{I_k}^2\prod_{t=0}^{r}n_t^{1-\delta_{I_k}^t}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(H_{n_\zeta}^{1-\delta_{I_k}^\zeta}\times(H_{n_\zeta}+K_{n_\zeta})^{\delta_{I_k}^\zeta}\right)\\
&\quad+\sum_{k=1}^{s}\sum_{J_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\sigma_{I_kJ_h}^2\prod_{t=0}^{r}n_t^{1-\delta_{I_kJ_h}^t}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(H_{n_\zeta}^{1-\delta_{I_kJ_h}^\zeta}\times(H_{n_\zeta}+K_{n_\zeta})^{\delta_{I_kJ_h}^\zeta}\right)\\
&\quad+\sigma_0^2\prod_{\zeta=0}^{r}\otimes(H_{n_\zeta}+K_{n_\zeta})\\
&=\sum_{k=1}^{s}\sum_{I_k\subset S}\sigma_{I_k}^2\prod_{t=0}^{r}n_t^{1-\delta_{I_k}^t}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes(H_{n_\zeta}+\delta_{I_k}^\zeta K_{n_\zeta})\\
&\quad+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\sigma_{I_kJ_h}^2\prod_{t=0}^{r}n_t^{1-\delta_{I_kJ_h}^t}H_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes(H_{n_\zeta}+\delta_{I_kJ_h}^\zeta K_{n_\zeta})\\
&\quad+\sigma_0^2\prod_{\zeta=0}^{r}\otimes(H_{n_\zeta}+K_{n_\zeta}),
\end{aligned}
$$

whose determinant is given by

$$
\begin{aligned}
(4.16)\quad & \left\{\sum_{\alpha=1}^{s}\sum_{M\alpha\subset S}\sum_{\beta=0}^{r-s}\sum_{N\beta\subset R-S}\sigma_{M\alpha N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{M\alpha N\beta}^\zeta}+\sigma_0^2\right\}\\
&\cdot\prod_{d=1}^{r-s}\prod_{U_d\subset R-S}\cdot\left\{\sum_{\alpha=1}^{s}\sum_{M\alpha\subset S}\sum_{\beta=d}^{r-s}\sum_{\substack{N\beta\supset U_d\\ N\beta\subset R-S}}\sigma_{M\alpha N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{M\alpha N\beta}^\zeta}+\sigma_0^2\right\}^{(n_{u_1}-1)\cdots(n_{u_d}-1)}\\
&\cdot\prod_{k=1}^{s}\prod_{I_k\subset S}\prod_{h=0}^{r-s}\prod_{J_h\subset R-S}\\
&\cdot\left\{\sum_{\alpha=k}^{s}\sum_{\substack{M\alpha\supset I_k\\ M\alpha\subset S}}\sum_{\beta=h}^{r-s}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-S}}\sigma_{M\alpha N\beta}^2\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{M\alpha N\beta}^\zeta}+\sigma_0^2\right\}^{(n_{i_1}-1)\cdots(n_{i_k}-1)(n_{j_1}-1)\cdots(n_{j_h}-1)}\\
&\cdot\left\{\sigma_0^2\right\}^{(n_0-1)n_1n_2\cdots n_r}.
\end{aligned}
$$

This formula is equivalent to (4.14).

4.2. The inverse of the variance matrix.

Before finding out the inverse of the variance matrix, we have to be prepared with some relations between the notations, defined in Definition 4.1, and some other results in order to simplify the complicated and tedious algebraic calculations.

LEMMA 4.1.

$$A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}=\frac{1}{n_{u_d}}\left[A_{(I_k)}^{(U_{d-1})}B_{(J_h)}^{(V_e)}-A_{(I_k u_d)}^{(U_{d-1})}B_{(J_h)}^{(V_e)}\right], \tag{4.17}$$

$$\begin{pmatrix} I_k\subset S;\ U_d\subset S;\ J_h\subset R-S;\ V_e\subset R-S; \\ k=0,\cdots,s;\ d=1,\cdots,s;\ e,h=0,\cdots,r-s. \end{pmatrix}.$$

PROOF.

$$\begin{aligned} A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)} &= \sum_{\beta=h}^{r-s-e}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-S-V_e}}\left[\sum_{\alpha=0}^{s-d+1}\sum_{\substack{M\alpha\subset S-U_{d-1}\\ M\alpha\supset I_k}}\sigma^2_{M\alpha N\beta}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^{\zeta}_{M\alpha N\beta U_d V_e}}\right. \\ &\qquad \left.-\sum_{\alpha=0}^{s-d+1}\sum_{\substack{M\alpha\supset(I_k u_d)\\ M\alpha\subset S-U_{d-1}}}\sigma^2_{M\alpha N\beta}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^{\zeta}_{M\alpha N\beta U_d V_e}}\right] \\ &= \frac{1}{n_{u_d}}\left[\sum_{\beta=h}^{r-s-e}\sum_{\substack{N\beta\supset J_h\\ N\beta\subset R-S-V_e}}\left\{\sum_{\alpha=0}^{s-d+1}\sum_{\substack{M\alpha\subset S-U_{d-1}\\ M\alpha\supset I_k}}\sigma^2_{M\alpha N\beta}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^{\zeta}_{M\alpha N\beta U_{d-1} V_e}}\right.\right. \\ &\qquad \left.\left.-\sum_{\alpha=0}^{s-d+1}\sum_{\substack{M\alpha\supset(I_k u_d)\\ M\alpha\subset S-U_{d-1}}}\sigma^2_{M\alpha N\beta}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^{\zeta}_{M\alpha N\beta U_{d-1} V_e}}\right\}\right] \\ &= \frac{1}{n_{u_d}}\left[A_{(I_k)}^{(U_{d-1})}B_{(J_h)}^{(V_e)}-A_{(I_k u_d)}^{(U_{d-1})}B_{(J_h)}^{(V_e)}\right]. \end{aligned} \tag{4.18}$$

The next result is stated without explicit proof.

LEMMA 4.2.

$$A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}=\frac{1}{n_{v_e}}\left[A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_{e-1})}-A_{(I_k)}^{(U_d)}B_{(J_h v_e)}^{(V_{e-1})}\right], \tag{4.19}$$

$$\begin{pmatrix} I_k\subset S;\ U_d\subset S;\ J_h\subset R-S;\ V_e\subset R-S; \\ k,d=0,\cdots,s;\ e=1,\cdots,r-s;\ h=0,\cdots,r-s. \end{pmatrix}.$$

From Lemma 4.1 and 4.2 we have

LEMMA 4.3.

$$A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}=\frac{1}{\prod_{\zeta=1}^{r}n_\zeta^{\delta^{\zeta}_{U_d}}}\left[\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^p A_{(I_k L_p)}B_{(J_h)}^{(V_e)}\right], \tag{4.20}$$

$$\begin{pmatrix} I_k\subset S;\ U_d\subset S;\ J_h\subset R-S;\ V_e\subset R-S; \\ k=0,\cdots,s;\ d=1,\cdots,s;\ e,h=0,\cdots,r-s. \end{pmatrix}$$

and

LEMMA 4.4.

$$A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}=\frac{1}{\prod_{\zeta=1}^{r}n_\zeta^{\delta^{\zeta}_{V_e}}}\left[\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^q A_{(I_k)}^{(U_d)}B_{(J_h T_q)}\right], \tag{4.21}$$

$$\begin{pmatrix} I_k\subset S;\ U_d\subset S;\ J_h\subset R-S;\ V_e\subset R-S; \\ k,d=0,\cdots,s;\ e=1,\cdots,r-s;\ h=0,\cdots,r-s. \end{pmatrix}.$$

These lemata can be proved easily by mathematical induction in the similar way to Lemma 4.2 in [2].

In the sequel we can express $A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}$ in terms of $A_{(I_kL_p)}B_{(J_hT_q)}$ using Lemma 4.3 and 4.4 in turn, which is given by

LEMMA 4.5.

$$A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}=\frac{1}{\prod_{\zeta=1}^{r} n_\zeta^{\delta_\zeta^{U_dV_e}}}\left[\sum_{p=0}^{d}\sum_{L_p\subset U_d}\sum_{q=0}^{e}\sum_{T_p\subset V_e}(-1)^{p+q}A_{(I_kL_p)}B_{(J_hT_q)}\right], \tag{4.22}$$

$$\begin{pmatrix} I_k\subset S;\ U_d\subset S;\ J_h\subset R-S;\ V_e\subset R-S; \\ k=0,\cdots,s;\ d=1,\cdots,s;\ h=0,\cdots,r-s;\ e=1,\cdots,r-s. \end{pmatrix}.$$

On the other hand we have

LEMMA 4.6. *Let G and E be the functions, which are finite and not vanished, defined on the subsets of the set of integers $(1,2,\cdots,r)$, then we have*

$$\sum_{d=1}^{k}\sum_{U_d\subset A_k}\left\{\sum_{\alpha=0}^{k-d}\sum_{M_\alpha\subset A_k-U_d}\frac{(-1)^\alpha}{G_{(U_dM_\alpha)}}\right\}\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^pE_{(L_p)}\right\} \tag{4.23}$$

$$=\sum_{d=1}^{k}\sum_{U_d\subset A_k}(-1)^d\left\{\frac{E_{(U_d)}-E}{G_{(U_d)}}\right\},$$

where E is the value of $E_{(U_d)}$ corresponding to $U_d=\phi$.

PROOF. Among the left side (4.23) the partial sum for $d+\alpha=c(c=1,2,\cdots,k)$ is given by

$$\sum_{d=1}^{c}\sum_{U_d\subset A_k}\sum_{M_{c-d}\subset A_k-U_d}\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^{c-d+p}\frac{E_{(L_p)}}{G_{(U_dM_{c-d})}}. \tag{4.24}$$

This is divided into three parts, the sum for $p=0$, the sum for $p=h(1\leqq h\leqq c-1)$ and the sum for $p=c$. And these are evaluated in (4.25), (4.26) and (4.27) respectively:

$$\begin{aligned} &\sum_{d=1}^{c}\sum_{U_d\subset A_k}\sum_{M_{c-d}\subset A_k-U_d}(-1)^{c-d}\frac{E}{G_{(U_dM_{c-d})}} \\ &=\sum_{U_c\subset A_k}(-1)^0\frac{E}{G_{(U_c)}}+\sum_{U_{c-1}\subset A_k}\sum_{M_1\subset A_k-U_{c-1}}(-1)^1\frac{E}{G_{(U_{c-1}M_1)}} \\ &+\sum_{U_{c-2}\subset A_k}\sum_{M_2\subset A_k-U_{c-2}}(-1)^2\frac{E}{G_{(U_{c-2}M_2)}}+\cdots\cdots+\sum_{U_1\subset A_k}\sum_{M_{c-1}\subset A_k-U_1}(-1)^{c-1}\frac{E}{G_{(U_1M_{c-1})}} \\ &=[{}_cC_0(-1)^0+{}_cC_1(-1)^1+\cdots+{}_cC_{c-1}(-1)^{c-1}]\sum_{U_c\subset A_k}\frac{E}{G_{(U_c)}} \\ &=-(-1)^c\sum_{U_c\subset A_k}\frac{E}{G_{(U_c)}}. \end{aligned} \tag{4.25}$$

$$\sum_{U_h\subset A_k}\sum_{M_{c-h}\subset A_k-U_h}\sum_{L_h\subset U_h}(-1)^c\frac{E_{(L_h)}}{G_{(U_hM_{c-h})}}+\sum_{U_{h+1}\subset A_k}\sum_{M_{c-h-1}\subset A_k-U_{h+1}}\sum_{L_h\subset U_{h+1}}\frac{(-1)^1E_{(L_h)}}{G_{(U_{h+1}M_{c-h-1})}} \tag{4.26}$$

$$+\cdots\cdots\cdots\cdots+\sum_{U_c\subset A_k}\sum_{L_h\subset U_c}\frac{(-1)^h E_{(L_h)}}{G_{(U_c)}}$$

$$=\left[\sum_{j=0}^{h}(-1)^{c-j}{}_{c-h}C_j\right]\sum_{U_h\subset A_k}\sum_{M_{c-h}\subset A_k-U_h}\sum_{L_h\subset U_h}\frac{E_{(L_h)}}{G_{(U_h M_{c-h})}}=0.$$

(4.27) $$\sum_{U_c\subset A_k}(-1)^c\frac{E_{(U_c)}}{G_{(U_c)}}.$$

Finally the sum of (4.24) with respect to c is equal to the right side of (4.23), which completes the proof.

Now we have made our necessary preparation, and we can enter into the calculation of the inversion of the variance matrix.

THEOLEM 4.2. *The inverse of the variance matrix* (4.1) *is given by*

(4.28) $$X_R E_{n_0}\otimes E_{n_1}\otimes\cdots\otimes E_{n_r}+\sum_{k=1}^{s}\sum_{I_k\subset S}X_{I_k}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta^\zeta_{I_k}}\times I_{n_\zeta}^{\delta^\zeta_{I_k}}\right)$$
$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}X_{I_kJ_h}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta^\zeta_{I_kJ_h}}\times I_{n_\zeta}^{\delta^\zeta_{I_kJ_h}}\right)$$
$$+\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}X_{J_h}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta^\zeta_{J_h}}\times I_{n_\zeta}^{\delta^\zeta_{J_h}}\right)+X_0 I_{n_0}\otimes I_{n_1}\otimes\cdots\otimes I_{n_r},$$

where

(4.29) $$X_0=\frac{1}{\sigma_0^2},$$

(4.30) $$X_R=\frac{1}{n_0}\left(\frac{1}{C_{(S)}D_{(R-S)}}-\frac{1}{\sigma_0^2}\right),$$

(4.31) $$X_{I_k,R-S}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{I_k,R-S}}}\left[\sum_{\alpha=0}^{s-k}\sum_{M_\alpha\subset S-I_k}\frac{(-1)^\alpha}{C_{(I_kM_\alpha)}D_{(R-S)}}\right],\qquad (I_k\subset S;\ k=1,\cdots,s),$$

(4.32) $$X_{S,J_h}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{S,J_h}}}\left[\sum_{\beta=0}^{r-s-h}\sum_{N_\beta\subset R-S-J_h}\frac{(-1)^\beta}{C_{(S)}D_{(J_hN_\beta)}}\right],\qquad (J_h\subset R-S;\ h=1,\cdots,r-s),$$

(4.33) $$X_{I_kJ_h}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{I_kJ_h}}}\left[\sum_{\alpha=0}^{s-k}\sum_{M_\alpha\subset S-I_k}\sum_{\beta=0}^{r-s-h}\sum_{N_\beta\subset R-S-J_h}\frac{(-1)^{\alpha+\beta}}{C_{(I_kM_\alpha)}D_{(J_hN_\beta)}}\right],$$
$$(I_k\subset S;\ J_h\subset R-S;\ k=1,\cdots,s;\ h=1,\cdots,r-s),$$

(4.34) $$X_{I_k}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta^\zeta_{I_k}}}\left[\sum_{\alpha=0}^{s-k}\sum_{M_\alpha\subset S-I_k}\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}\frac{(-1)^{\alpha+\beta}}{C_{(I_kM_\alpha)}D_{(N_\beta)}}\right],\qquad (I_k\subset S;\ k=1,\cdots,s),$$

$$X_{J_h}=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{J_h}^{\zeta}}}\Bigg[\sum_{\alpha=1}^{s}\sum_{M_\alpha\subset S}\sum_{\beta=0}^{r-s-h}\sum_{N\beta\subset R-S-J_h}\frac{(-1)^{\alpha+\beta}}{C_{(M_\alpha)}D_{(J_hN\beta)}} \tag{4.35}$$

$$+\sum_{\beta=0}^{r-s-h}\sum_{N\beta\subset R-S-J_h}\frac{(-1)^{\beta}}{E_{(J_hN\beta)}}\Bigg], \qquad (J_h\subset R-S;\ h=1,\cdots,r-s),$$

$$X_G=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta}\Bigg[\frac{1}{G}+\sum_{\beta=1}^{r-s}\sum_{N\beta\subset R-S}\frac{(-1)^{\beta}}{G_{(N\beta)}}+\sum_{\alpha=1}^{s}\sum_{M_\alpha\subset S}\sum_{\beta=0}^{r-s}\sum_{N\beta\subset R-S}\frac{(-1)^{\alpha+\beta}}{C_{(M_\alpha)}D_{(N\beta)}}\Bigg]. \tag{4.36}$$

PROOF. Anticipating the inverse to be the form of (4.28), we search for the condition that (4.28) is actually the inverse.

The product of the variance matrix (4.1) and the matrix (4.28) is

$$E_{n_0}\otimes E_{n_1}\otimes\cdots\otimes E_{n_r} \tag{4.37}$$

$$\cdot\Bigg[X_G\{CD\}+\sum_{k=1}^{s}\sum_{I_k\subset S}X_{I_k}\{A^{(I_k)}B\}$$

$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}X_{I_kJ_h}\{A^{(I_k)}B^{(J_h)}\}+\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}X_{J_h}\{AB^{(J_h)}\}\Bigg]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_S^{\zeta}}\times I_{n_\zeta}^{\delta_S^{\zeta}}\right)$$

$$\cdot\Bigg[X_S\{C_{(S)}D\}+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{SV_e}\{A_{(S)}B^{(V_e)}\}+X_0\sigma_S^2\Bigg]$$

$$+\sum_{k=1}^{s-1}\sum_{I_k\subset S}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{I_k}^{\zeta}}\times I_{n_\zeta}^{\delta_{I_k}^{\zeta}}\right)$$

$$\cdot\Bigg[X_{I_k}\{C_{(I_k)}D\}+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}X_{I_kU_d}\{A_{(I_k)}^{(U_d)}B\}$$

$$+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{I_kU_dV_e}\{A_{(I_k)}^{(U_d)}B^{(V_e)}\}$$

$$+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{I_kV_e}\{A_{(I_k)}B^{(V_e)}\}+X_0\sigma_{I_k}^2\Bigg]$$

$$+E_{n_0}\otimes I_{n_1}\otimes I_{n_2}\otimes\cdots\otimes I_{n_r}$$

$$\cdot\Bigg[X_R\{C_{(S)}D_{(R-S)}\}+X_0\sigma_R^2\Bigg]$$

$$+\sum_{k=1}^{s-1}\sum_{I_k\subset S}E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta_{I_k,R-S}^{\zeta}}\times I_{n_\zeta}^{\delta_{I_k,R-S}^{\zeta}}\right)$$

$$\cdot\Bigg[X_{I_k,R-S}\{C_{(I_k)}D_{(R-S)}\}+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}X_{I_kU_dR-S}\{A_{(I_k)}^{(U_d)}B_{(R-S)}\}+X_0\sigma_{I_kR-S}^2\Bigg]$$

$$+\sum_{h=1}^{r-s-1}\sum_{J_h\subset R-S} E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta^\zeta_{SJ_h}}\times I_{n_\zeta}^{\delta^\zeta_{SJ_h}}\right)$$

$$\cdot\left[X_{SJ_h}\{C_{(S)}D_{(J_h)}\}+\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{SJ_hV_e}\{A_{(S)}B_{(J_h)}^{(V_e)}\}+X_0\sigma^2_{SJ_h}\right]$$

$$+\sum_{k=1}^{s-1}\sum_{I_k\subset S}\sum_{h=1}^{r-s-1}\sum_{J_h\subset R-S} E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta^\zeta_{I_kJ_h}}\times I_{n_\zeta}^{\delta^\zeta_{I_kJ_h}}\right)$$

$$\cdot\Big[X_{I_kJ_h}\{C_{(I_k)}D_{(J_h)}\}+\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{I_kJ_hV_e}\{A_{(I_k)}B_{(J_h)}^{(V_e)}\}$$

$$+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k} X_{I_kU_dJ_h}\{A_{(I_k)}^{(U_d)}B_{(J_h)}\}$$

$$+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{I_kU_dJ_hV_e}\{A_{(I_k)}^{(U_d)}B_{(J_h)}^{(V_e)}\}+X_0\sigma^2_{I_kJ_h}\Big]$$

$$+E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta^\zeta_{R-S}}\times I_{n_\zeta}^{\delta^\zeta_{R-S}}\right)$$

$$\cdot\left[X_{R-S}\{CD_{(R-S)}\}+\sum_{d=1}^{s}\sum_{U_d\subset S} X_{U_dR-S}\{A^{(U_d)}B_{(R-S)}\}\right]$$

$$+\sum_{h=1}^{r-s-1}\sum_{J_h\subset R-S} E_{n_0}\otimes\prod_{\zeta=1}^{r}\otimes\left(E_{n_\zeta}^{1-\delta^\zeta_{J_h}}\times I_{n_\zeta}^{\delta^\zeta_{J_h}}\right)$$

$$\cdot\Big[X_{J_h}\{CD_{(J_h)}\}+\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{J_hV_e}\{AB_{(J_h)}^{(V_e)}\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S} X_{U_dJ_h}\{A^{(U_d)}B_{(J_h)}\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{U_dJ_hV_e}\{A^{(U_d)}B_{(J_h)}^{(V_e)}\}\Big]$$

$$+I_{n_0}\otimes I_{n_1}\otimes I_{n_2}\otimes\cdots\otimes I_{n_r}X_0\sigma_0^2.$$

Thus the condition is expressed by the following equations,

(4.38) $$X_0\sigma_0^2=1,$$

(4.39) $$X_R\{C_{(S)}D_{(R-S)}\}=-X_0\sigma_R^2,$$

(4.40) $$X_{I_kR-S}\{C_{(I_k)}D_{(R-S)}\}=-\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k} X_{I_kU_dR-S}\{A_{(I_k)}^{(U_d)}B_{(R-S)}\}-X_0\sigma^2_{I_kR-S},$$
$$(I_k\subset S;\ k=1,\cdots,s),$$

(4.41) $$X_{SJ_h}\{C_{(S)}D_{(J_h)}\}=-\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{SJ_hV_e}\{A_{(S)}B_{(J_h)}^{(V_e)}\}-X_0\sigma^2_{SJ_h},$$
$$(J_h\subset R-S;\ h=1,\cdots,r-s),$$

(4.42) $$X_{I_kJ_h}\{C_{(I_k)}D_{(J_h)}\}$$

$$= -\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{I_kJ_hV_e}\{A_{(I_k)}B^{(V_e)}_{(J_h)}\} - \sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k} X_{I_kU_dJ_h}\{A^{(U_d)}_{(I_k)}B_{(J_h)}\}$$

$$-\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{I_kU_dJ_hV_e}\{A^{(U_d)}_{(I_k)}B^{(V_e)}_{(J_h)}\} - X_0\sigma^2_{I_kJ_h},$$

$$\begin{pmatrix} I_k\subset S;\ J_h\subset R-S \\ k=1,\cdots,s-1;\ h=1,\cdots,r-s-1 \end{pmatrix},$$

(4.43) $$X_S\{C_{(S)}D\} = -\sum_{e=1}^{r-s}\sum_{V_e\subset R-S} X_{SV_e}\{A_{(S)}B^{(V_e)}\} - X_0\sigma^2_S,$$

(4.44) $$X_{R-S}\{CD_{(R-S)}\} = -\sum_{d=1}^{s}\sum_{U_d\subset S} X_{U_dR-S}\{A^{(U_d)}B_{(R-S)}\},$$

(4.45) $$X_{I_k}\{C_{(I_k)}D\} = -\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k} X_{I_kU_d}\{A^{(U_d)}_{(I_k)}B\} - \sum_{e=1}^{r-s}\sum_{V_e\subset R-S} X_{I_kV_e}\{A_{(I_k)}B^{(V_e)}\}$$

$$-\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S} X_{I_kU_dV_e}\{A^{(U_d)}_{(I_k)}B^{(V_e)}\} - X_0\sigma^2_{I_k},$$

$$(I_k\subset S;\ k=1,\cdots,s-1),$$

(4.46) $$X_{J_h}\{CD_{(J_h)}\}$$

$$= -\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{J_hV_e}\{AB^{(V_e)}_{(J_h)}\} - \sum_{d=1}^{s}\sum_{U_d\subset S} X_{U_dJ_h}\{A^{(U_d)}B_{(J_h)}\}$$

$$-\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s-h}\sum_{V_e\subset R-S-J_h} X_{U_dJ_hV_e}\{A^{(U_d)}B^{(V_e)}_{(J_h)}\},$$

$$(J_h\subset R-S;\ h=1,\cdots,r-s-1),$$

(4.47) $$X_G\{CD\} = -\sum_{k=1}^{s}\sum_{I_k\subset S} X_{I_k}\{A^{(I_k)}B\} - \sum_{h=1}^{r-s}\sum_{J_h\subset R-S} X_{J_h}\{AB^{(J_h)}\}$$

$$-\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\sum_{k=1}^{s}\sum_{I_k\subset S} X_{I_kJ_h}\{A^{(I_k)}B^{(J_h)}\}.$$

Now the proof of this theorem is completed by proving the following:

LEMMA 4.6. *The solutions of the equations* (4.38), $\cdots$, (4.47) *are given by* (4.29), $\cdots$, (4.36).

PROOF. (4.29) comes from (4.38) directly and (4.30) comes from (4.29) and (4.39).

(i) Solutions for X_{I_kR-S}, $(I_k\subset S;\ k=1,2,\cdots,s-1)$.

(4.31) is obtained by mathematical induction in k in (4.40) and from (4.29) and (4.30), which is as follows.

At the first stage, we shall prove (4.31) holds true for all $I_{s-1}\subset S$. The equation to be solved is

(4.48) $$X_{I_{s-1}R-S}\{C_{(I_{s-1})}D_{(R-S)}\} = -\left[X_R\{A^{(S-I_{s-1})}_{(I_{s-1})}B_{(R-S)}\} + X_0\sigma^2_{I_{s-1}R-S}\right]$$

$$= -\frac{1}{n_0}\left[\frac{1}{C_{(S)}D_{(R-S)}}\{A^{(S-I_{s-1})}_{(I_{s-1})}B_{(R-S)}\}\right]$$

$$=\frac{1}{n_0 n_{i_s}}\left[\frac{1}{C_{(S)}D_{(R-S)}}\{A_{(S)}B_{(R-S)}-A_{(I_s-1)}B_{(R-S)}\}\right]$$

$$=\frac{1}{n_0 n_{i_s}}\left[\frac{1}{C_{(S)}D_{(R-S)}}\{C_{(S)}D_{(R-S)}-C_{(I_s-1)}D_{(R-S)}\}\right].$$

Hence we have

$$X_{I_s-1R-S}=\frac{1}{n_0 n_{i_s}}\left[\frac{1}{C_{(I_s-1)}D_{(R-S)}}-\frac{1}{C_{(S)}D_{(R-S)}}\right], \tag{4.49}$$

which completes the first stage.

At the second stage, we shall prove, assuming that this holds true for all $I_k\subset S$ when $k=q,\ q+1,\cdots,\ s-1$, this also holds true for all $I_{q-1}\subset S$. Under this assumptions we obtain by Lemma 4.5 and 4.6

$$\begin{aligned}
(4.50)\qquad & X_{I_{q-1}R-S}\{C_{(I_{q-1})}D_{(R-S)}\}\\
&=-\left[\sum_{d=1}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}X_{I_{q-1}U_dR-S}\{A_{(I_{q-1})}^{(U_d)}B_{(R-S)}\}+X_0\sigma^2_{I_{q-1}R-S}\right]\\
&=-\left[\sum_{d=1}^{s-q}\sum_{U_d\subset S-I_{q-1}}X_{I_{q-1}U_dR-S}\{A_{(I_{q-1})}^{(U_d)}B_{(R-S)}\}+X_R\{A_{(I_{q-1})}^{(S-I_{q-1})}B_{(R-S)}\}\right.\\
&\qquad\left.+\frac{X_0}{n_0}A_{(I_{q-1})}^{(S-I_{q-1})}B_{(R-S)}\right]\\
&=-\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^{\zeta}_{I_{q-1}R-S}}}\left[\sum_{d=1}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}\left\{\sum_{\alpha=0}^{s-q-d+1}\sum_{M_\alpha\subset S-(I_{q-1}\cup U_d)}\frac{(-1)^{\alpha}}{C_{(I_{q-1}U_dM_\alpha)}D_{(R-S)}}\right\}\right.\\
&\qquad\left.\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^pA_{(I_{q-1}L_p)}B_{(R-S)}\right\}\right]\\
&=-\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^{\zeta}_{I_{q-1}R-S}}}\left[\sum_{d=1}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}(-1)^{d}\left\{\frac{A_{(I_{q-1}U_d)}B_{(R-S)}-A_{(I_{q-1})}B_{(R-S)}}{C_{(I_{q-1}U_d)}D_{(R-S)}}\right\}\right]\\
&=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^{\zeta}_{I_{q-1}R-S}}}\left[\sum_{d=1}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}(-1)^{d}\left\{\frac{C_{(I_{q-1})}D_{(R-S)}-C_{(I_{q-1}U_d)}D_{(R-S)}}{C_{(I_{q-1}U_d)}D_{(R-S)}}\right\}\right].
\end{aligned}$$

Hence we have

$$\begin{aligned}
(4.51)\qquad X_{I_{q-1}R-S}&=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^{\zeta}_{I_{q-1}R-S}}}\left[\sum_{d=1}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}\frac{(-1)^d}{C_{(I_{q-1}U_d)}D_{(R-S)}}-\sum_{d=1}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}\frac{(-1)^d}{C_{(I_{q-1})}D_{(R-S)}}\right]\\
&=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^{\zeta}_{I_{q-1}R-S}}}\left[\sum_{d=0}^{s-q+1}\sum_{U_d\subset S-I_{q-1}}\frac{(-1)^d}{C_{(I_{q-1}U_d)}D_{(R-S)}}\right].
\end{aligned}$$

(ii) Solutions for X_{SJ_h}, $(J_h\subset R-S;\ h=1,2,\cdots,r-s-1)$.

(4.32) is obtained by mathematical induction in h in (4.41) and from (4.29) and (4.30), which is as follows. At the first stage, we shall prove (4.32) holds true for all $J_{r-s-1}\subset R-S$. The equation to be solved is

$$
\begin{aligned}
(4.52)\qquad X_{SJ_{r-s-1}}\{C_{(S)}D_{(J_{r-s-1})}\} &= -\left[X_R\{A_{(S)}B_{(J_{r-s-1})}^{(R-S-J_{r-s-1})}\}+X_0\sigma^2_{SJ_{r-s-1}}\right] \\
&= -\frac{1}{n_0}\left[\frac{A_{(S)}B_{(J_{r-s-1})}^{(R-S-J_{r-s-1})}}{C_{(S)}D_{(R-S)}}\right] = \frac{1}{n_0 n_{j_{r-s-1}}}\left[\frac{A_{(S)}B_{(R-S)}-A_{(S)}B_{(J_{r-s-1})}}{C_{(S)}D_{(R-S)}}\right] \\
&= \frac{1}{n_0 n_{j_{r-s-1}}}\left[\frac{C_{(S)}D_{(R-S)}-C_{(S)}D_{(J_{r-s-1})}}{C_{(S)}D_{(R-S)}}\right].
\end{aligned}
$$

And we have

$$
(4.53)\qquad X_{SJ_{r-s-1}} = \frac{1}{n_0 n_{j_{r-s-1}}}\left[\frac{1}{C_{(S)}D_{(J_{r-s-1})}}-\frac{1}{C_{(S)}D_{(R-S)}}\right].
$$

At the second stage, we shall prove, assuming that this holds true for all $J_h\subset R-S$ when $h=c,\ c+1,\cdots,\ r-s-1$, this also holds true for all $J_{c-1}\subset R-S$. Under these assumptions, by Lemma 4.5 and 4.6 the equation to be solved is given as follows

$$
\begin{aligned}
(4.54)\qquad & X_{SJ_{c-1}}\{C_{(S)}D_{(J_{c-1})}\} \\
&= -\left[\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}} X_{SJ_{c-1}V_e}\{A_{(S)}B_{(J_{c-1})}^{(V_e)}\}+X_0\sigma^2_{SJ_{c-1}}\right] \\
&= -\frac{1}{N_{SJ_{c-1}}}\left[\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\left\{\sum_{\beta=0}^{r-s-c+1-e}\sum_{N\beta\subset R-S-(J_{c-1}\cup V_e)}\frac{(-1)^\beta}{C_{(S)}D_{(J_{c-1}V_eN\beta)}}\right\}\right. \\
&\qquad \left.\cdot\left\{\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^q A_{(S)}B_{(J_{c-1},T_q)}\right\}\right] \\
&= -\frac{1}{N_{SJ_{c-1}}}\left[\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}(-1)^e\left\{\frac{A_{(S)}B_{(J_{c-1}V_e)}-A_{(S)}B_{(J_{c-1})}}{C_{(S)}D_{(J_{c-1}V_e)}}\right\}\right] \\
&= \frac{1}{N_{SJ_{c-1}}}\left[\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}(-1)^e\left\{\frac{C_{(S)}D_{(R-S)}-C_{(S)}D_{(J_{c-1}V_e)}}{C_{(S)}D_{(J_{c-1}V_e)}}\right\}\right],
\end{aligned}
$$

where $N_{SJ_{c-1}}=\prod_{\zeta}^{r} n_\zeta^{1-\delta^\zeta_{SJ_{c-1}}}$.

Then we have

$$
\begin{aligned}
(4.55)\qquad X_{SJ_{c-1}} &= \frac{1}{N_{SJ_{c-1}}}\left[\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\frac{(-1)^e}{C_{(S)}D_{(J_{c-1}V_e)}}-\sum_{e=1}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\frac{(-1)^e}{C_{(S)}D_{(J_{c-1})}}\right] \\
&= \frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{SJ_{c-1}}}}\left[\sum_{e=0}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\frac{(-1)^e}{C_{(S)}D_{(J_{c-1}V_e)}}\right],
\end{aligned}
$$

which completes the second stage.

In the following cases we shall make use of Lemma 4.5 and 4.6 without refering to them explicitly.

(**iii**) Solutions for $X_{I_kJ_h}$, $(I_k\subset S;\ J_h\subset R-S;\ k=1,2,\cdots,s-1;\ h=1,2,\cdots,r-s-1)$.

(4.33) is obtained by mathematical induction in $k+h$ in (4.42) and from (4.29), (4.30), (4.31) and (4.32).

At the first stage we shall prove (4.33) holds true for all $I_{s-1}\subset S$ and $J_{r-s-1}\subset R-S$. The equation to be solved is

$$
\begin{aligned}
(4.56)\qquad & X_{I_{s-1}J_{r-s-1}}\{C_{(I_{s-1})}D_{(J_{r-s-1})}\} \\
&= -\Big[X_{I_{s-1}R-S}\{A_{(I_{s-1})}B_{(J_{r-s-1})}^{(R-S-J_{r-s-1})}\} + X_{SJ_{r-s-1}}\{A_{(I_{s-1})}^{(S-I_{s-1})}B_{(J_{r-s-1})}\} \\
&\qquad + X_R\{A_{(I_{s-1})}^{(S-I_{s-1})}B_{(J_{r-s-1})}^{(R-S-J_{r-s-1})}\} + X_0\sigma^2_{I_{s-1}J_{r-s-1}}\Big] \\
&= \frac{-1}{n_0n_{i_s}n_{j_{r-s}}}\Big[\frac{A_{(I_{s-1})}B_{(J_{r-s-1})}-A_{(I_{s-1})}B_{(R-S)}}{C_{(I_{s-1})}D_{(R-S)}} \\
&\qquad + \frac{A_{(I_{s-1})}B_{(J_{r-s-1})}-A_{(S)}B_{(J_{r-s-1})}}{C_{(S)}D_{(J_{r-s-1})}} + \frac{A_{(S)}B_{(R-S)}-A_{(I_{s-1})}B_{(J_{r-s-1})}}{C_{(S)}D_{(R-S)}}\Big] \\
&= \frac{1}{n_0n_{i_s}n_{j_{r-s}}}\Big[\frac{C_{(I_{s-1})}D_{(R-S)}-C_{(I_{s-1})}D_{(J_{r-s-1})}}{C_{(I_{s-1})}D_{(R-S)}} \\
&\qquad + \frac{C_{(S)}D_{(J_{r-s-1})}-C_{(I_{s-1})}D_{(J_{r-s-1})}}{C_{(S)}D_{(J_{r-s-1})}} + \frac{C_{(I_{s-1})}D_{(J_{r-s-1})}-C_{(S)}D_{(R-S)}}{C_{(S)}D_{(R-S)}}\Big].
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
(4.57)\qquad X_{I_{s-1}J_{r-s-1}} &= \frac{1}{n_0n_{i_s}n_{j_{r-s}}}\Big[\frac{1}{C_{(I_{s-1})}D_{(J_{r-s-1})}} - \frac{1}{C_{(I_{s-1})}D_{(R-S)}} \\
&\qquad - \frac{1}{C_{(S)}D_{(J_{r-s-1})}} + \frac{1}{C_{(S)}D_{(R-S)}}\Big],
\end{aligned}
$$

which completes the first stage.

At the second stage, we shall prove. assuming that this holds true for all $I_k\subset S$ and J_{c-k} when $k=1,2,\cdots,c-1$, this also holds true for all $I_k\subset S$ nad J_{c-1-k} when $k=1, 2,\cdots,c-2$. Then we observe

$$
\begin{aligned}
(4.58)\qquad & X_{I_kJ_{c-1-k}}\{C_{(I_k)}D_{(J_{c-1-k})}\} \\
&= -\Big[\sum_{e=1}^{r-s-c+1+k}\ \sum_{V_e\subset R-S-J_{c-1-k}} X_{I_kJ_{c-1-k}V_e}\{A_{(I_k)}B_{(J_{c-1-k})}^{(V_e)}\} \\
&\qquad + \sum_{d=1}^{s-k}\ \sum_{U_d\subset S-I_k} X_{I_kU_dJ_{c-1-k}}\{A_{(I_k)}^{(U_d)}B_{(J_{c-1-k})}\} \\
&\qquad + \sum_{d=1}^{s-k}\ \sum_{U_d\subset S-I_k}\ \sum_{e=1}^{r-s-c+1+k}\ \sum_{V_e\subset R-S-J_{c-1-k}} X_{I_kU_dJ_{c-1-k}V_e}\{A_{(I_k)}^{(U_d)}B_{(J_{c-1-k})}^{(V_e)}\} + X_0\sigma^2_{I_kJ_{c-1-k}}\Big]
\end{aligned}
$$

$$
\begin{aligned}
=\frac{-1}{N_{I_kJ_{c-1-k}}}\Bigg[&\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\left\{\sum_{\alpha=0}^{s-k-d}\sum_{M_\alpha\subset S-(I_k\cup U_d)}\sum_{\beta=0}^{r-s-c+k+1}\sum_{N_\beta\subset R-S-J_{c-1-k}}\frac{(-1)^{\alpha+\beta}}{C_{(I_kU_dM_\alpha)}D_{(J_{c-1-k}N_\beta)}}\right\}\\
&\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^pA_{(I_kL_p)}B_{(J_{c-1-k})}\right\}\\
&+\sum_{e=1}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}\left\{\sum_{\alpha=0}^{s-k}\sum_{M_\alpha\subset S-I_k}\sum_{\beta=0}^{r-s-c+k+1-e}\sum_{N_\beta\subset R-S-(J_{c-1-k}\cup V_e)}\frac{(-1)^{\alpha+\beta}}{C_{(I_kM_\alpha)}D_{(J_{c-1-k}V_eN_\beta)}}\right\}\\
&\cdot\left\{\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^qA_{(I_k)}B_{(J_{c-1-k}T_q)}\right\}\\
&+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}\left\{\sum_{\alpha=0}^{s-k-d}\sum_{M_\alpha\subset S-(I_k\cup U_d)}\right.\\
&\cdot\left.\sum_{\beta=0}^{r-s-c+k+1-e}\sum_{N_\beta\subset R-S-(J_{c-1-k}\cup V_e)}\frac{(-1)^{\alpha+\beta}}{C_{(I_kU_dM_\alpha)}D_{(J_{c-1-k}V_eN_\beta)}}\right\}\\
&\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^{p+q}A_{(I_kL_p)}B_{(J_{c-1-k}T_q)}\right\}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
=\frac{-1}{N_{I_kJ_{c-1-k}}}\Bigg[&\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}(-1)^d\left\{\frac{A_{(I_kU_d)}B_{(J_{c-1-k})}-A_{(I_k)}B_{(J_{c-1-k})}}{C_{(I_kU_d)}D_{(J_{c-1-k})}}\right\}\\
&+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}(-1)^{d+e}\left\{\frac{A_{(I_kU_d)}B_{(J_{c-1-k})}-A_{(I_k)}B_{(J_{c-1-k})}}{C_{(I_kU_d)}D_{(J_{c-1-k}V_e)}}\right\}\\
&+\sum_{e=1}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}(-1)^e\left\{\frac{A_{(I_k)}B_{(J_{c-1-k}V_e)}-A_{(I_k)}B_{(J_{c-1-k})}}{C_{(I_k)}D_{(J_{c-1-k}V_e)}}\right\}\\
&+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}(-1)^{d+e}\left\{\frac{A_{(I_k)}B_{(J_{c-1-k}V_e)}-A_{(I_k)}B_{(J_{c-1-k})}}{C_{(I_kU_d)}D_{(J_{c-1-k}V_e)}}\right\}\\
&+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}(-1)^{d+e}\\
&\cdot\left\{\frac{A_{(I_kU_d)}B_{(J_{c-1-k}V_e)}-A_{(I_k)}B_{(J_{c-1-k}V_e)}-A_{(I_kU_d)}B_{(J_{c-1-k})}+A_{(I_k)}B_{(J_{c-1-k})}}{C_{(I_kU_d)}D_{(J_{c-1-k}V_e)}}\right\}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
=\frac{1}{N_{I_kJ_{c-1-k}}}\Bigg[&\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}(-1)^d\left\{\frac{C_{(I_k)}D_{(J_{c-1-k})}-C_{(I_kU_d)}D_{(J_{c-1-k})}}{C_{(I_kU_d)}D_{(J_{c-1-k})}}\right\}\\
&+\sum_{e=1}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}(-1)^e\left\{\frac{C_{(I_k)}D_{(J_{c-1-k})}-C_{(I_k)}D_{(J_{c-1-k}V_e)}}{C_{(I_k)}D_{(J_{c-1-k}V_e)}}\right\}\\
&+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}(-1)^{d+e}\left\{\frac{C_{(I_k)}D_{(J_{c-1-k})}-C_{(I_kU_d)}D_{(J_{c-1-k}V_e)}}{C_{(I_kU_d)}D_{(J_{c-1-k}V_e)}}\right\}\Bigg]
\end{aligned}
$$

where $N_{I_kJ_{c-1-k}}=\prod_{\zeta}^{r} n_{\zeta}$ [illegible].

And so we have

(4.59)
$$X_{I_kJ_{c-1-k}}=\frac{1}{N_{I_kJ_{c-1-k}}}\left[\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}(-1)^d\left\{\frac{1}{C_{(I_kU_d)}D_{(J_{c-1-k})}}-\frac{1}{C_{(I_k)}D_{(J_{c-1-k})}}\right\}\right.$$
$$+\sum_{e=1}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}(-1)^e\left\{\frac{1}{C_{(I_k)}D_{(J_{c-1-k}V_e)}}-\frac{1}{C_{(I_k)}D_{(J_{c-1-k})}}\right\}$$
$$\left.+\sum_{d=1}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=1}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}(-1)^{d+e}\left\{\frac{1}{C_{(I_kU_d)}D_{(J_{c-1-k}V_e)}}-\frac{1}{C_{(I_k)}D_{(J_{c-1-k})}}\right\}\right]$$
$$=\frac{1}{N_{I_kJ_{c-1-k}}}\left[\sum_{d=0}^{s-k}\sum_{U_d\subset S-I_k}\sum_{e=0}^{r-s-c+k+1}\sum_{V_e\subset R-S-J_{c-1-k}}\frac{(-1)^{d+e}}{C_{(I_kU_d)}D_{(J_{c-1-k}V_e)}}\right].$$

(iv) Solution for X_S.

The solution for X_S is obtained by substituting (4.29), (4.30) and (4.32) in (4.43):

(4.60)
$$X_S\{C_{(S)}D\}=-\left[\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{SV_e}\{A_{(S)}B^{(V_e)}\}+X_S\sigma_S^2\right]$$
$$=\frac{-1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_S^\zeta}}\left[\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\left\{\sum_{\beta=0}^{r-s-e}\sum_{N\beta\subset R-S-V_e}\frac{(-1)^\beta}{C_{(S)}D_{(V_eN\beta)}}\right\}\left\{\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^qA_{(S)}B_{(T_q)}\right\}\right]$$
$$=\frac{1}{N_S}\left[\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^e\left\{\frac{A_{(S)}B-A_{(S)}B_{(V_e)}}{C_{(S)}D_{(V_e)}}\right\}\right]$$
$$=\frac{1}{N_S}\left[\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^e\left\{\frac{C_{(S)}D-C_{(S)}D_{(V_e)}}{C_{(S)}D_{(V_e)}}\right\}\right],$$

which gives us

(4.61)
$$X_S=\frac{1}{N_S}\left[\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^e\left\{\frac{1}{C_{(S)}D_{(V_e)}}-\frac{1}{C_{(S)}D}\right\}\right]$$
$$=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_S^\zeta}}\left[\sum_{e=0}^{r-s}\sum_{V_e\subset R-S}\frac{(-1)^e}{C_{(S)}D_{(V_e)}}\right].$$

(v) Solution for X_{R-S}.

The solution for X_{R-S} is obtained by inserting (4.29), (4.30) and (4.31) in (4.44):

(4.62)
$$X_{R-S}\{CD_{(R-S)}\}=-\left[\sum_{d=1}^{s}\sum_{U_d\subset S}X_{U_dR-S}\{A^{(U_d)}B_{(R-S)}\}\right]$$
$$=\frac{-1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{R-S}^\zeta}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}\left\{\sum_{\alpha=0}^{s-d}\sum_{M_\alpha\subset S-U_d}\frac{(-1)^\alpha}{C_{(U_dM_\alpha)}D_{(R-S)}}\right\}\right.$$
$$\left.\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^pA_{(L_p)}B_{(R-S)}\right\}\right]$$
$$=\frac{-1}{N_{R-S}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{A_{(U_d)}B_{(R-S)}-AB_{(R-S)}}{C_{(U_d)}D_{(R-S)}}\right\}\right]$$

$$=\frac{1}{N_{R-S}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{CD_{(R-S)}-C_{(U_d)}D_{(R-S)}}{C_{(U_d)}D_{(R-S)}}\right\}\right].$$

On the other hand it holds that

$$CD_{(J_h)}=G_{(J_h)}, \qquad (J_h\subset R-S;\ h=0.1,\cdots,r-s), \tag{4.63}$$

since

$$C^{(S)}D_{(J_h)}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_S^\zeta}}\left[CD_{(J_h)}+\sum_{p=1}^{s}\sum_{L_p\subset S}(-1)^pC_{(L_p)}D_{(J_h)}\right]=0, \tag{4.64}$$

and so

$$CD_{(J_h)}=\sum_{p=1}^{s}\sum_{L_p\subset S}(-1)^{p-1}C_{(L_p)}D_{(J_h)}=G_{(J_h)}. \tag{4.65}$$

Therefore we have

$$\begin{aligned} X_{R-S}&=\frac{1}{N_{R-S}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{1}{C_{(U_d)}D_{(R-S)}}-\frac{1}{G_{(R-S)}}\right\}\right]\\ &=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{R-S}^\zeta}}\left[\sum_{d=1}^{s}\sum_{U_d\subset S}\frac{(-1)^d}{C_{(U_d)}D_{(R-S)}}+\frac{1}{G_{(R-S)}}\right]. \end{aligned} \tag{4.66}$$

(**vi**) Solutions for X_{I_k}, $(I_k\subset S,\ k=1,2,\cdots,s-1)$.

(4.34) is obtained by mathematical induction in k in (4.45) and from (4.29), (4.30), (4.31), (4.32), (4.33) and (4.61).

At the first stage we shall prove (4.34) holds true for all $I_{s-1}\subset S$. The equation to be solved is

$$\begin{aligned} X_{I_{s-1}}\{C_{(I_{s-1})}D\}=&-\left[X_S\{A_{(I_{s-1})}^{(S-I_{s-1})}B\}+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{I_{s-1}V_e}\{A_{(I_{s-1})}B^{(V_e)}\}\right.\\ &\left.+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{SV_e}\{A_{(I_{s-1})}^{(S-I_{s-1})}B^{(V_e)}\}+X_0\sigma_{I_{s-1}}^2\right]\\ =&\frac{1}{\prod_{\zeta}^{r}n_\zeta^{1-\delta_{I_{s-1}}^\zeta}}\left[\left\{\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}\frac{(-1)^\beta}{C_{(S)}D_{(N_\beta)}}\right\}\{A_{(I_{s-1})}B-A_{(S)}B\}\right.\\ &+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\left\{\sum_{\alpha=0}^{1}\sum_{M_\alpha\subset S-I_{s-1}}\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^{\alpha+\beta}}{C_{(I_{s-1}M_\alpha)}D_{(V_eN_\beta)}}\right\}\\ &\cdot\left\{\sum_{\eta=1}^{e}\sum_{T_\eta\subset V_e}(-1)^\eta A_{(I_{s-1})}B_{(T_\eta)}\right\}\\ &+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\left\{\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^\beta}{C_{(S)}D_{(V_eN_\beta)}}\right\} \end{aligned} \tag{4.67}$$

$$\cdot\left\{\sum_{p=0}^{1}\sum_{L_p\subset S-I_{s-1}}\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^{p+q}A_{(I_{s-1}L_p)}B_{(T_q)}\right\}\Bigg]$$

$$=\frac{-1}{N_{I_{s-1}}}\left[\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}(-1)^{\beta}\left\{\frac{A_{(I_{s-1})}B-A_{(S)}B}{C_{(S)}D_{(N_\beta)}}\right\}\right.$$

$$+\sum_{\alpha=0}^{1}\sum_{M_\alpha\subset S-I_{s-1}}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{\alpha+e}\left\{\frac{A_{(I_{s-1})}B_{(V_e)}-A_{(I_{s-1})}B}{C_{(I_{s-1}M_\alpha)}D_{(V_e)}}\right\}$$

$$\left.+\sum_{p=0}^{1}\sum_{L_p\subset S-I_{s-1}}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{p+e}\left\{\frac{A_{(I_{s-1}L_p)}B_{(V_e)}-A_{(I_{s-1}L_p)}B}{C_{(S)}D_{(V_e)}}\right\}\right]$$

$$=\frac{-1}{N_{I_{s-1}}}\left[\frac{A_{(I_{s-1})}B-A_{(S)}B}{C_{(S)}D}\right.$$

$$\left.+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{e}\left\{\frac{A_{(I_{s-1})}B-A_{(S)}B_{(V_e)}}{C_{(S)}D_{(V_e)}}+\frac{A_{(I_{s-1})}B_{(V_e)}-A_{(I_{s-1})}B}{C_{(I_{s-1})}D_{(V_e)}}\right\}\right]$$

$$=\frac{1}{N_{I_{s-1}}}\left[\frac{C_{(S)}D-C_{(I_{s-1})}D}{C_{(S)}D}\right.$$

$$\left.+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{e}\left\{\frac{C_{(S)}D_{(V_e)}-C_{(I_{s-1})}D}{C_{(S)}D_{(V_e)}}+\frac{C_{(I_{s-1})}D-C_{(I_{s-1})}D_{(V_e)}}{C_{(I_{s-1})}D_{(V_e)}}\right\}\right].$$

Then we have

$$X_{I_{s-1}}=\frac{1}{\prod_{\zeta}^{r} n_\zeta^{1-\delta^{\zeta}_{I_{s-1}}}}\left[\frac{1}{C_{(I_{s-1})}D}-\frac{1}{C_{(S)}D}\right. \tag{4.68}$$

$$\left.+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-)^{e}\left\{\frac{1}{C_{(I_{s-1})}D_{(V_e)}}-\frac{1}{C_{(S)}D_{(V_e)}}\right\}\right],$$

which completes the first stage.

At the second stage we shall prove, assuming that this holds true for all $I_k\subset S$ when $k=c,\ c+1,\cdots,s-1$, that (4.29), (4.30), (4.31), (4.32), (4.33), and (4.61) hold, this also holds true for all $I_{c-1}\subset S$. The equation is to be given by

$$X_{I_{c-1}}\{C_{(I_{c-1})}D\} \tag{4.69}$$

$$=-\left[\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}X_{I_{c-1}U_d}\{A^{(U_d)}_{(I_{c-1})}B\}+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{I_{c-1}V_e}\{A_{(I_{c-1})}B^{(V_e)}\}\right.$$

$$\left.+\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{I_{c-1}U_dV_e}\{A^{(U_d)}_{(I_{c-1})}B^{(V_e)}\}+X\sigma_{I_{c-1}}\right]$$

$$=\frac{-1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^{\zeta}_{I_{c-1}}}}\left[\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}\left\{\sum_{\alpha=0}^{s-c+1-d}\sum_{M_\alpha\subset S-(I_{c-1}\cup U_d)}\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}\frac{(-1)^{\alpha+\beta}}{C_{(I_{c-1}U_dM_\alpha)}D_{(N_\beta)}}\right\}\right.$$

$$\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^{p}A_{(I_{c-1}L_p)}B\right\}$$

$$
+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\left\{\sum_{\alpha=0}^{s-c+1}\sum_{M_\alpha\subset S-I_{c-1}}\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^{\alpha+\beta}}{C_{(I_{c-1}M_\alpha)}D_{(V_eN_\beta)}}\right\}
$$

$$
\cdot\left\{\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^q A_{(I_{c-1})}B_{(T_q)}\right\}
$$

$$
+\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}
$$

$$
\cdot\left\{\sum_{\alpha=0}^{s-c+1-d}\sum_{M_\alpha\subset S-(I_{c-1}\cup U_d)}\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^{\alpha+\beta}}{C_{(I_{c-1}U_dM_\alpha)}D_{(V_eN_\beta)}}\right\}
$$

$$
\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^{p+q}A_{(I_{c-1}L_p)}B_{(T_q)}\right\}\Bigg]
$$

$$
=\frac{1}{N_{I_{c-1}}}\left[\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}(-1)^d\left\{\frac{C_{(I_{c-1})}D-C_{(I_{c-1}U_d)}D}{C_{(I_{c-1}U_d)}D}\right\}\right.
$$

$$
+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^e\left\{\frac{C_{(I_{c-1})}D-C_{(I_{c-1})}D_{(V_e)}}{C_{(I_{c-1})}D_{(V_e)}}\right\}
$$

$$
\left.+\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{d+e}\left\{\frac{C_{(I_{c-1})}D-C_{(I_{c-1}U_d)}D_{(V_e)}}{C_{(I_{c-1}U_d)}D_{(V_e)}}\right\}\right].
$$

Hence we have

$$
\tag{4.70} X_{I_{c-1}}=\frac{1}{N_{I_{c-1}}}\left[\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}\frac{(-1)^d}{C_{(I_{c-1}U_d)}D}+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\frac{(-1)^e}{C_{(I_{c-1})}D_{(V_e)}}\right.
$$

$$
\left.+\sum_{d=1}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\frac{(-1)^{d+e}}{C_{(I_{c-1}U_d)}D_{(V_e)}}+\frac{1}{C_{(I_{c-1})}D}\right]
$$

$$
=\frac{1}{\prod_{\zeta=1}^{r}n_\zeta^{1-\delta^\zeta_{I_{c-1}}}}\left[\sum_{d=0}^{s-c+1}\sum_{U_d\subset S-I_{c-1}}\sum_{e=0}^{r-s}\sum_{V_e\subset R-S}\frac{(-1)^{d+e}}{C_{(I_{c-1}U_d)}D_{(V_e)}}\right],
$$

which completes the second stage.

(**vii**) Solutions for X_h, $(J_h\subset R-S;\ h=1, 2, \cdots, r-s-1)$.

(4.35) is obtained by mathematical induction in h in (4.46) and from (4.30), (4.31), (4.32), (4.33) and (4.66).

At the first stage, we shall prove (4.35) holds true for all $J_{r-s-1}\subset R-S$. The equation to be solved is

$$
\tag{4.71} X_{J_{r-s-1}}\{G_{(J_{r-s-1})}\}
$$

$$
=-\left[X_{R-S}\{AB_{(J_{r-s-1})}-AB_{(R-S)}\}+\sum_{d=1}^{s}\sum_{U_d\subset S}X_{U_dJ_{r-s-1}}\{A^{(U_d)}B_{(J_{r-s-1})}\}\right.
$$

$$
\left.+\sum_{d=1}^{s}\sum_{U_d\subset S}X_{U_dR-S}\{A^{(U_d)}B_{(J_{r-s-1})}-A^{(U_d)}B_{(R-S)}\}\right]
$$

$$=\frac{-1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{J_r-s-1}}}\Bigg[\Bigg\{\sum_{d=1}^{s}\sum_{U_d\subset S}\frac{(-1)^d}{C_{(U_d)}D_{(R-S)}}+\frac{1}{G_{(R-S)}}\Bigg\}\{AB_{(J_r-s-1)}-AB_{(R-S)}\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\Bigg\{\sum_{\alpha=0}^{s-d}\sum_{M_\alpha\subset S-U_d}\sum_{\beta=0}^{1}\sum_{N\beta\subset R-S-J_r-s-1}\frac{(-1)^{\alpha+\beta}}{C_{(U_dM_\alpha)}D_{(J_r-s-1N\beta)}}\Bigg\}$$

$$\cdot\Bigg\{\sum_{p=0}^{d}\sum_{Lp\subset U_d}(-1)^pA_{(Lp)}B_{(J_r-s-1)}\Bigg\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\Bigg\{\sum_{\alpha=0}^{s-d}\sum_{M_\alpha\subset S-U_d}\frac{(-1)^\alpha}{C_{(U_dM_\alpha)}D_{(R-S)}}\Bigg\}$$

$$\cdot\Bigg\{\sum_{p=0}^{d}\sum_{Lp\subset U_d}\sum_{q=0}^{1}\sum_{Tq\subset R-S-J_r-s-1}(-1)^{p+q}A_{(Lp)}B_{(J_r-s-1Tq)}\Bigg\}\Bigg]$$

$$=\frac{-1}{N_{J_r-s-1}}\Bigg[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\Bigg\{\frac{AB_{(J_r-s-1)}-AB_{(R-S)}}{C_{(U_d)}D_{(R-S)}}\Bigg\}+\frac{AB_{(J_r-s-1)}-AB_{(R-S)}}{G_{(R-S)}}$$

$$+\sum_{\beta=0}^{1}\sum_{N\beta\subset R-S-J_r-s-1}\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^{\beta+d}\Bigg\{\frac{A_{(U_d)}B_{(J_r-s-1)}-AB_{(J_r-s-1)}}{C_{(U_d)}D_{(J_r-s-1N\beta)}}\Bigg\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\Bigg\{\frac{A_{(U_d)}B_{(J_r-s-1)}-AB_{(J_r-s-1)}}{C_{(U_d)}D_{(R-S)}}\Bigg\}$$

$$-\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\Bigg\{\frac{A_{(U_d)}B_{(R-S)}-AB_{(R-S)}}{C_{(U_d)}D_{(R-S)}}\Bigg\}\Bigg]$$

$$=\frac{1}{N_{J_r-s-1}}\Bigg[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\Bigg\{\frac{C_{(U_d)}D_{(R-S)}-CD_{(J_r-s-1)}}{C_{(U_d)}D_{(R-S)}}$$

$$+\frac{CD_{(J_r-s-1)}-C_{(U_d)}D_{(J_r-s-1)}}{C_{(U_d)}D_{(J_r-s-1)}}\Bigg\}+\frac{CD_{(R-S)}-CD_{(J_r-s-1)}}{G_{(R-S)}}\Bigg]$$

$$=\frac{1}{N_{J_r-s-1}}\Bigg[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\Bigg\{\frac{C_{(U_d)}D_{(R-S)}-G_{(J_r-s-1)}}{C_{(U_d)}D_{(R-S)}}$$

$$+\frac{G_{(J_r-s-1)}-C_{(U_d)}D_{(J_r-s-1)}}{C_{(U_d)}D_{(J_r-s-1)}}\Bigg\}+\frac{G_{(R-S)}-G_{(J_r-s-1)}}{G_{(R-S)}}\Bigg].$$

And so we have

$$X_{J_r-s-1}=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{J_r-s-1}}}\Bigg[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\Bigg\{\frac{1}{C_{(U_d)}D_{(J_r-s-1)}}-\frac{1}{C_{(U_d)}D_{(R-S)}}\Bigg\}$$
$$+\frac{1}{G_{(J_r-s-1)}}-\frac{1}{G_{(R-S)}}\Bigg]. \tag{4.72}$$

At the second stage we shall prove, assuming that this holds true for all J_r-R-S when $h=c,\ c+1,\cdots,r-s-1$, that (4.30), (4.31), (4.32), (4.33) and (4.66) hold, this also

holds true for all $J_{c-1} \subset R-S$. The equation to be solved is given by

$$
\begin{aligned}
(4.73)\qquad & X_{J_{c-1}}\{G_{(J_{c-1})}\} \\
&= -\Bigg[\sum_{e=1}^{r-s-c+1} \sum_{V_e \subset R-S-J_{c-1}} X_{J_{c-1}V_e}\{AB^{(V_e)}_{(J_{c-1})}\} + \sum_{d=1}^{s} \sum_{U_d \subset S} X_{U_d J_{c-1}}\{A^{(U_d)} B_{(J_{c-1})}\} \\
&\qquad + \sum_{d=1}^{s} \sum_{U_d \subset S} \sum_{e=1}^{r-s-c+1} \sum_{V_e \subset R-S-J_{c-1}} X_{U_d J_{c-1} V_e}\{A^{(U_d)} B^{(V_e)}_{(J_{c-1})}\}\Bigg] \\
&= \frac{-1}{\prod n_i - N_{J_{c-1}}} \Bigg[\sum_{e=1}^{r-s-c+1} \sum_{V_e \subset R-S-J_{c-1}} \Bigg\{\sum_{\alpha=1}^{s} \sum_{M_\alpha \subset S} \sum_{\beta=0}^{r-s-c+1-e} \sum_{N_\beta \subset R-S-(J_{c-1} \cup V_e)} \\
&\qquad \cdot \frac{(-1)^{\alpha+\beta}}{C_{(M_\alpha)} D_{(J_{c-1} V_e N_\beta)}} + \sum_{\beta=0}^{r-s-c+1-e} \sum_{N_\beta \subset R-S-(J_{c-1} \cup V_e)} \frac{(-1)^{\beta}}{G_{(J_{c-1} V_e N_\beta)}}\Bigg\} \\
&\qquad \cdot \Bigg\{\sum_{q=0}^{e} \sum_{T_q \subset V_e} (-1)^q AB_{(J_{c-1} T_q)}\Bigg\} \\
&\qquad + \sum_{d=1}^{s} \sum_{U_d \subset S} \Bigg\{\sum_{\alpha=0}^{s-d} \sum_{M_\alpha \subset S-U_d} \sum_{\beta=0}^{r-s-c+1} \sum_{N_\beta \subset R-S-J_{c-1}} \frac{(-1)^{\alpha+\beta}}{C_{(U_d M_\alpha)} D_{(J_{c-1} N_\beta)}}\Bigg\} \\
&\qquad \cdot \Bigg\{\sum_{p=0}^{d} \sum_{L_p \subset U_d} (-1)^p A_{(L_p)} B_{(J_{c-1})}\Bigg\} \\
&\qquad + \sum_{d=1}^{s} \sum_{U_d \subset S} \sum_{e=1}^{r-s-c+1} \sum_{V_e \subset R-S-J_{c-1}} \Bigg\{\sum_{\alpha=0}^{s-d} \sum_{M_\alpha \subset S-U_d} \sum_{\beta=0}^{r-s-c+1-e} \sum_{N_\beta \subset R-S-(J_{c-1} \cup V_e)} \\
&\qquad \cdot \frac{(-1)^{\alpha+\beta}}{C_{(U_d M_\alpha)} D_{(J_{c-1} V_e N_\beta)}}\Bigg\}\Bigg\{\sum_{p=0}^{d} \sum_{L_p \subset U_d} \sum_{q=0}^{e} \sum_{T_q \subset V_e} (-1)^{p+q} A_{(L_p)} B_{(J_{c-1} T_q)}\Bigg\}\Bigg] \\
&= \frac{-1}{N_{J_{c-1}}} \Bigg[\sum_{d=1}^{s} \sum_{U_d \subset S} (-1)^d \Bigg\{\frac{A_{(U_d)} B_{(J_{c-1})} - AB_{(J_{c-1})}}{C_{(U_d)} D_{(J_{c-1})}}\Bigg\} \\
&\qquad + \sum_{d=1}^{s} \sum_{U_d \subset S} \sum_{e=1}^{r-s-c+1} \sum_{V_e \subset R-S-J_{c-1}} (-1)^{d+e} \Bigg\{\frac{A_{(U_d)} B_{(J_{c-1} V_e)} - AB_{(J_{c-1})}}{C_{(U_d)} D_{(J_{c-1} V_e)}}\Bigg\} \\
&\qquad + \sum_{e=1}^{r-s-c+1} \sum_{V_e \subset R-S-J_{c-1}} (-1)^e \Bigg\{\frac{AB_{(J_{c-1} V_e)} - AB_{(J_{c-1})}}{G_{(J_{c-1} V_e)}}\Bigg\}\Bigg] \\
&= \frac{1}{N_{J_{c-1}}} \Bigg[\sum_{d=1}^{s} \sum_{U_d \subset S} (-1)^d \Bigg\{\frac{G_{(J_{c-1})} - C_{(U_d)} D_{(J_{c-1})}}{C_{(U_d)} D_{(J_{c-1})}}\Bigg\} \\
&\qquad + \sum_{d=1}^{s} \sum_{U_d \subset S} \sum_{e=1}^{r-s-c+1} \sum_{V_e \subset R-S-J_{c-1}} (-1)^{d+e} \Bigg\{\frac{G_{(J_{c-1})} - C_{(U_d)} D_{(J_{c-1} V_e)}}{C_{(U_d)} D_{(J_{c-1} V_e)}}\Bigg\} \\
&\qquad + \sum_{e=1}^{r-s-c+1} \sum_{V_e \subset R-S-J_{c-1}} (-1)^e \Bigg\{\frac{G_{(J_{c-1})} - G_{(J_{c-1} V_e)}}{G_{(J_{c-1} V_e)}}\Bigg\}\Bigg].
\end{aligned}
$$

Therefore we have

(4.74) $$X_{J_{c-1}}=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{J_{c-1}}}}\Bigg[\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=0}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\frac{(-1)^{d+e}}{C_{(U_d)}D_{(J_{c-1}V_e)}}$$

$$+\sum_{e=0}^{r-s-c+1}\sum_{V_e\subset R-S-J_{c-1}}\frac{(-1)^e}{G_{(J_{c-1}V_e)}}\Bigg].$$

(**viii**) Solution for X_G.

(4.36) is obtained by inserting (4.30), …, (4.35), (4.61) and (4.66) in (4.47). After inserting them in (4.47), we have

(4.75) $$X_G\{G\}=-\Bigg[\sum_{d=1}^{s}\sum_{U_d\subset S}X_{U_d}\{A^{(U_d)}B\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{U_dV_e}\{A^{(U_d)}B^{(V_e)}\}+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}X_{V_e}\{AB^{(V_e)}\}\Bigg]$$

$$=\frac{-1}{\prod_{\zeta=0}^{r} n_\zeta}\Bigg[\sum_{d=1}^{s}\sum_{U_d\subset S}\left\{\sum_{\alpha=0}^{s-d}\sum_{M_\alpha\subset S-U_d}\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}\frac{(-1)^{\alpha+\beta}}{C_{(U_dM_\alpha)}D_{(N_\beta)}}\right\}$$

$$\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}(-1)^pA_{(L_p)}B\right\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\left\{\sum_{\alpha=0}^{s-d}\sum_{M_\alpha\subset S-U_d}\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^{\alpha+\beta}}{C_{(U_dM_\alpha)}D_{(V_eN_\beta)}}\right\}$$

$$\cdot\left\{\sum_{p=0}^{d}\sum_{L_p\subset U_d}\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^{p+q}A_{(L_p)}B_{(T_q)}\right\}$$

$$+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\left\{\sum_{\alpha=1}^{s}\sum_{M_\alpha\subset S}\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^{\alpha+\beta}}{C_{(M_\alpha)}D_{(V_eN_\beta)}}\right.$$

$$\left.+\sum_{\beta=0}^{r-s-e}\sum_{N_\beta\subset R-S-V_e}\frac{(-1)^\beta}{G_{(V_eN_\beta)}}\right\}\left\{\sum_{q=0}^{e}\sum_{T_q\subset V_e}(-1)^qAB_{(T_q)}\right\}$$

$$=\frac{-1}{\prod_{\zeta=0}^{r} n_\zeta}\Bigg[\sum_{\beta=0}^{r-s}\sum_{N_\beta\subset R-S}\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^{\beta+d}\left\{\frac{A_{(U_d)}B-AB}{C_{(U_d)}D_{(N_\beta)}}\right\}$$

$$+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{d+e}\left\{\frac{A_{(U_d)}B_{(V_e)}-AB_{(V_e)}}{C_{(U_d)}D_{(V_e)}}\right\}$$

$$-\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{d+e}\left\{\frac{A_{(U_d)}B-AB}{C_{(U_d)}D_{(V_e)}}\right\}$$

$$+\sum_{\alpha=1}^{s}\sum_{M_\alpha\subset S}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{\alpha+e}\left\{\frac{AB_{(V_e)}-AB}{C_{(M_\alpha)}D_{(V_e)}}\right\}$$

$$+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^e\left\{\frac{AB_{(V_e)}-AB}{G_{(V_e)}}\right\}\Bigg]$$

$$
= \frac{1}{\prod_{\zeta=0}^{r} n_\zeta}\Bigg[\sum_{d=1}^{s}\sum_{U_d\subset S}(-1)^d\left\{\frac{G-C_{(U_d)}D}{C_{(U_d)}D}\right\}
$$

$$
+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^{d+e}\left\{\frac{G-C_{(U_d)}D_{(V_e)}}{C_{(U_d)}D_{(V_e)}}\right\}
$$

$$
+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}(-1)^e\left\{\frac{G-G_{(V_e)}}{G_{(V_e)}}\right\}\Bigg].
$$

Hence we have

$$
X_G=\frac{1}{\prod_{\zeta=0}^{r} n_\zeta}\Bigg[\frac{1}{G}+\sum_{e=1}^{r-s}\sum_{V_e\subset R-S}\frac{(-1)^e}{G_{(V_e)}}
\tag{4.76}
$$

$$
+\sum_{d=1}^{s}\sum_{U_d\subset S}\sum_{e=0}^{r-s}\sum_{V_e\subset R-S}\frac{(-1)^{d+e}}{C_{(U_d)}D_{(V_e)}}\Bigg].
$$

Finally the proof of Lemma 4.6 is completed, and also Theorem 4.2 is proved.

4.3. The joint density function.

The joint density function in Type II is given in the following,

THEOREM 4.3. *The joint density function of all observations in our case is given by*

$$
f(X)=(2\pi)^{-n_0n_1\cdots n_r/2}G^{-1/2}\prod_{d=1}^{r-s}\prod_{U_d\subset R-S}\{G_{(U_d)}\}^{-(n_{u_1}-1)\cdots(n_{u_d}-1)/2}
\tag{4.77}
$$

$$
\cdot\prod_{k=1}^{s}\prod_{I_k\subset S}\prod_{h=0}^{r-s}\prod_{J_h\subset R-S}\{C_{(I_k)}D_{(J_h)}\}^{-(n_{i_1}-1)\cdots(n_{i_k}-1)(n_{j_1}-1)\cdots(n_{j_h}-1)/2}
$$

$$
\cdot\{\sigma_0^2\}^{-(n_0-1)n_1n_2\cdots n_r/2}
$$

$$
\cdot\, exp\Bigg[-\frac{1}{2}\Bigg\{\prod_{j=0}^{r} n_j(\bar{X}-\mu)^2\frac{1}{G}+\sum_{d=1}^{r-s}\sum_{U_d\subset R-S}\frac{T_{(U_d)}}{G_{(U_d)}}
$$

$$
+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=0}^{r-s}\sum_{J_h\subset R-S}\frac{S_{(I_kJ_h)}}{C_{(I_k)}D_{(J_h)}}+\frac{S_0}{\sigma_0^2}\Bigg\}\Bigg],
$$

where

$$
T_{(U_d)}=\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{U_d}^{\zeta}}
\tag{4.78}
$$

$$
\cdot\sum_{t_{u_1},\cdots,t_{u_d}}\left\{\sum_{\beta=0}^{d}\sum_{L_\beta\subset U_d}(-1)^{d-\beta}\bar{X}_{t_{l_1}t_{l_2}\cdots t_{l_\beta}}-\alpha(u_1,\cdots,u_d;\ t_{u_1},\cdots,t_{u_d})\right\}^2,
$$

$$
(U_d\subset R-S;\ d=1,2,\cdots,r-s),
$$

$$
S_{(I_kJ_h)}=\prod_{\zeta=0}^{r} n_\zeta^{1-\delta_{I_kJ_h}^{\zeta}}\sum_{t_{i_1},\cdots,t_{i_k}}\sum_{t_{j_1},\cdots,t_{j_h}}\left\{\sum_{\beta=0}^{k+h}\sum_{L_\beta\subset(I_k\cup J_h)}(-1)^{k+h-\beta}\bar{X}_{t_{l_1}\cdots t_{l_\beta}}\right\}^2,
\tag{4.79}
$$

$$
(I_k\subset S;\ J_h\subset R-S;\ k=1,\cdots,s;\ h=0,\cdots,r-s),
$$

(4.80) $$S_0=\sum_{t_0,t_1,\cdots,t_r}(x_{t_0t_1\cdots t_r}-\bar{X}_{t_1t_2\cdots t_r})^2,$$

and

(4.81) $$\bar{X}_{t_{l_1}t_{l_2}\cdots t_{l_\beta}}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{L_\beta}^\zeta}}\sum_{\substack{t_{v_1},\cdots,t_{v_{r-\beta}}\\ V_{r-\beta}\subset R-L_\beta}}\sum_{t_0}x_{t_0t_1\cdots t_r},\qquad (L_\beta\subset R,\ \beta=0,\cdots,r),$$

where in (4.81) $\bar{X}_{t_{l_1}\cdots t_{l_\beta}}=\bar{X}$ *when* $\beta=0$.

PROOF. As it is obvious that the type of the density function is the normal distribution, the constant factor in (4.77) is easily derived from Theorem 4.1, and there remains only to derive the quadratic form of $x_{t_0t_1\cdots t_r}$.

Now, let us introduce new variables defined by

(4.82) $$y_{t_0t_1\cdots t_r}=x_{t_0t_1\cdots t_r}-\mu-\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h}),$$

(4.83) $$Y_{t_{l_1}t_{l_2}\cdots t_{l_\beta}}=\sum_{\substack{t_{v_1},\cdots,t_{v_{r-\beta}}\\ V_{r-\beta}\subset R-L_\beta}}\sum_{t_0}y_{t_0t_1\cdots t_r},\qquad (L_\beta\subset R;\ \beta=0,\cdots,r),$$

and

(4.84) $$\bar{Y}_{t_{l_1}t_{l_2}\cdots t_{l_\beta}}=\frac{1}{\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{L_\beta}^\zeta}}Y_{t_{l_1}t_{l_2}\cdots t_{l_\beta}},\qquad (L_\beta\subset R;\ \beta=0,\cdots,r),$$

and when $\beta=0$ we shall use the same convention for $Y_{t_{l_1}t_{l_2}\cdots t_{l_\beta}}$ and $\bar{Y}_{t_{l_1}t_{l_2}\cdots t_{l_\beta}}$ as the one stated in this theorem. Furthermore, we note that (4.84) is expressed in terms of $\bar{X}_{t_{l_1}\cdots t_{l_\beta}}$ and $\alpha(v_1,\cdots,v_p;\ t_{v_1},\cdots,t_{v_p})$ by the assumptions (2.2), as follows

(4.85) $$\bar{Y}_{t_{i_1}\cdots t_{i_k}t_{j_1}\cdots t_{j_h}}=\bar{X}_{t_{i_1}\cdots t_{i_k}t_{j_1}\cdots t_{j_h}}-\mu-\sum_{p=1}^{h}\sum_{V_p\subset J_h}\alpha(v_1,\cdots,v_p;\ t_{v_1},\cdots,t_{v_p}),$$
$$(I_k\subset S;\ J_h\subset R-S;\ k=0,\cdots,s;h=1,\cdots,r-s),$$

(4.86) $$\bar{Y}_{t_{i_1}\cdots t_{i_k}}=\bar{X}_{t_{i_1}\cdots t_{i_k}}-\mu,\qquad (I_k\subset S;\ k=0,\cdots,s).$$

Using the inverse matrix derived in Theorem 4.2 we have the quadratic form in the joint density function:

(4.87) $$S=X_G\Big(\sum_{t_0,t_1,\cdots,t_r}y_{t_0t_1\cdots t_r}\Big)^2$$
$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=0}^{r-s}\sum_{J_h\subset R-S}X_{I_kJ_h}$$
$$\cdot\left\{\sum_{t_{i_1},\cdots,t_{i_k}}\sum_{t_{j_1},\cdots,t_{j_h}}\Big(\sum_{\substack{t_{p_1},\cdots,t_{p_{s-k}}\\ P_{s-k}\subset S-I_k}}\sum_{\substack{t_{q_1},\cdots,t_{q_{r-s-h}}\\ Q_{r-s-h}\subset R-S-J_h}}\sum_{t_0}y_{t_0t_1\cdots t_r}\Big)^2\right\}$$
$$+\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}X_{J_h}\left\{\sum_{t_{j_1},\cdots,t_{j_h}}\Big(\sum_{\substack{t_{q_1},\cdots,t_{q_{r-s-h}}\\ Q_{r-s-h}\subset R-S-J_h}}\sum_{t_0,t_1,\cdots,t_s}y_{t_0t_1\cdots t_s}\Big)^2\right\}$$

$$+X_0 \sum_{t_0, t_1, \cdots, t_r} y^2_{t_0 t_1 \cdots t_r}$$

$$= \frac{1}{\prod_{\zeta=0}^{r} n_\zeta} \left[\frac{1}{G} + \sum_{\beta=1}^{r-s} \sum_{N\beta \subset R-S} \frac{(-1)^\beta}{G_{(N\beta)}} + \sum_{\alpha=1}^{s} \sum_{M\alpha \subset S} \sum_{\beta=0}^{r-s} \sum_{N\beta \subset R-S} \frac{(-1)^{\alpha+\beta}}{C_{(M\alpha)} D_{(N\beta)}} \right] Y^2$$

$$+ \sum_{k=1}^{s} \sum_{I_k \subset S} \sum_{h=0}^{r-s} \sum_{J_h \subset R-S} \frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{I_k J_h}}}$$

$$\cdot \left[\sum_{\alpha=0}^{s-k} \sum_{M\alpha \subset S-I_k} \sum_{\beta=0}^{r-s-h} \sum_{N\beta \subset R-S-J_h} \frac{(-1)^{\alpha+\beta}}{C_{(I_k M\alpha)} D_{(J_h N\beta)}} \right]$$

$$\cdot \left(\sum_{t_{i_1}, \cdots, t_{i_k}} \sum_{t_{j_1}, \cdots, t_{j_h}} Y^2_{t_{i_1} \cdots t_{i_k} t_{j_1} \cdots t_{j_h}} \right)$$

$$+ \sum_{h=1}^{r-s} \sum_{J_h \subset R-S} \frac{1}{\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{J_h}}} \left[\sum_{\alpha=1}^{s} \sum_{M\alpha \subset S} \sum_{\beta=0}^{r-s-h} \sum_{N\beta \subset R-S-J_h} \frac{(-1)^{\alpha+\beta}}{C_{(M\alpha)} D_{(J_h N\beta)}} \right.$$

$$\left. + \sum_{\beta=0}^{r-s-h} \sum_{N\beta \subset R-S-J_h} \frac{(-1)^\beta}{G_{(J_h N\beta)}} \right] \left(\sum_{t_{j_1}, \cdots, t_{j_h}} Y^2_{t_{j_1} \cdots t_{j_h}} \right)$$

$$+ \frac{1}{\sigma_0^2} \left(\sum_{t_0 t_1 \cdots t_r} y^2_{t_0 t_1 \cdots t_r} \right).$$

The calculation of (4.87) follows the similar line to that of (3.43), which is given by

(4.88)
$$S = \prod_{\zeta=0}^{r} n_\zeta \, \bar{Y}^2 \frac{1}{G}$$

$$+ \sum_{h=1}^{r-s} \sum_{J_h \subset R-S} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{J_h}} \sum_{t_{j_1}, \cdots, t_{j_h}} \left\{ \sum_{\beta=0}^{h} \sum_{L\beta \subset J_h} (-1)^{h-\beta} \bar{Y}_{t_{l_1} \cdots t_{l_\beta}} \right\}^2 \frac{1}{G_{(J_h)}}$$

$$+ \sum_{k=1}^{s} \sum_{I_k \subset S} \sum_{h=1}^{r-s} \sum_{J_h \subset R-S} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{I_k J_h}}$$

$$\cdot \sum_{t_{i_1}, \cdots, t_{i_k}} \sum_{t_{j_1}, \cdots, t_{j_h}} \left\{ \sum_{\beta=0}^{k+h} \sum_{L\beta \subset (I_k \cup J_h)} (-1)^{k+h-\beta} \bar{Y}_{t_{l_1} \cdots t_{l_\beta}} \right\}^2 \frac{1}{C_{(I_k)} D_{(J_h)}}$$

$$+ \sum_{k=1}^{s} \sum_{I_k \subset S} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{I_k}} \sum_{t_{i_1}, \cdots, t_{i_k}} \left\{ \sum_{\beta=0}^{k} \sum_{L\beta \subset I_k} (-1)^{k-\beta} \bar{Y}_{t_{l_1} \cdots t_{l_\beta}} \right\}^2 \frac{1}{C_{(I_k)} D}$$

$$+ \sum_{t_0, t_1, \cdots t_r} \left(y_{t_0 t_1 \cdots t_r} - \bar{Y}_{t_1 t_2 \cdots t_r} \right)^2 \frac{1}{\sigma_0^2}.$$

In virtue of (4.85) and (4.86), (4.88) is equal to

(4.89)
$$\prod_{\zeta=0}^{r} n_\zeta (\bar{X} - \mu)^2 \frac{1}{G}$$

$$+ \sum_{h=1}^{r-s} \sum_{J_h \subset R-S} \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{J_h}} \sum_{t_{j_1}, \cdots, t_{j_h}} \frac{1}{G_{(J_h)}}$$

$$\cdot\left[\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\left\{\bar{X}_{t_{l_1}\cdots t_{l_\beta}}-\mu-\sum_{p=1}^{\beta}\sum_{V_p\subset L_\beta}\alpha(v_1,\cdots,v_p;\ t_{v_1},\cdots,t_{v_p})\right\}\right]^2$$

$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=1}^{r-s}\sum_{J_h\subset R-S}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{I_kJ_h}^{\zeta}}\sum_{t_{i_1},\cdots,t_{i_k}}\sum_{t_{j_1},\cdots,t_{j_h}}$$

$$\cdot\left[\sum_{\lambda=0}^{k}\sum_{A_\lambda\subset I_k}\sum_{\nu=0}^{h}\sum_{B_\nu\subset J_h}(-1)^{k+h-\lambda-\nu}\left\{\bar{X}_{t_{a_1}\cdots t_{a_\lambda}t_{b_1}\cdots t_{b_\nu}}-\mu\right.\right.$$

$$\left.\left.-\sum_{p=1}^{\nu}\sum_{V_p\subset B_\nu}\alpha(v_1,\cdots,v_p;\ t_{v_1},\cdots,t_{v_p})\right\}\right]^2\frac{1}{C_{(I_k)}D_{(J_h)}}$$

$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{I_k}^{\zeta}}\sum_{t_{i_1},\cdots,t_{i_k}}\left[\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\{\bar{X}_{t_{l_1}\cdots t_{l_\beta}}-\mu\}\right]^2\frac{1}{C_{(I_k)}D}$$

$$+\sum_{t_0,t_1,\cdots,t_r}(x_{t_0t_1\cdots t_r}-\bar{X}_{t_1t_2\cdots t_r})^2\frac{1}{\sigma_0^2}.$$

The three formulas squared in the second, third and forth terms are simplified in (4.90), (4.91) and (4.92), respectively.

(4.90) $$\sum_{\beta=0}^{h}\sum_{L_\beta\subset J_h}(-1)^{h-\beta}\bar{X}_{t_{l_1}\cdots t_{l_\beta}}-\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h}).$$

(4.91) $$\sum_{\lambda=0}^{k}\sum_{A_\lambda\subset I_k}\sum_{\nu=0}^{h}\sum_{B_\nu\subset J_h}(-1)^{k+h-\lambda-\nu}\bar{X}_{t_{a_1}\cdots t_{a_\lambda}t_{b_1}\cdots t_{b_\nu}}$$

$$-\sum_{\lambda=0}^{k}\sum_{A_\lambda\subset I_k}(-1)^{k-\lambda}\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})$$

$$=\sum_{\beta=0}^{k+h}\sum_{L_\beta\subset(I_k\cup J_h)}(-1)^{k+h}\bar{X}_{t_{l_1}\cdots t_{l_\beta}}.$$

(4.92) $$\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\bar{X}_{t_{l_1}\cdots t_{l_\beta}}.$$

Finally, as (4.89) is equal to the quadratic formula in (4.77), the theorem is proved.

4.4. Estimation.

Also in this type we need to have the analogue of Lemma 3.3:

LEMMA 4.7. *The joint density function of* $x_{t_0t_1\cdots t_r}$, (4.77), *is equal to*

(4.93) $$f(X)=K\varphi_{(\sigma^2,\alpha,\mu)}$$

$$\cdot\ exp\left[\prod_{j=0}^{r}n_jZ^{(1)}\frac{\mu}{G}\right.$$

$$+\sum_{d=1}^{r-s}\sum_{U_d\subset R-S}\sum_{t_{u_1}=1}^{n_{u_1}-1}\sum_{t_{u_2}=1}^{n_{u_2}-1}\cdots\sum_{t_{u_d}=1}^{n_{u_d}-1}\prod_{\zeta=0}^{r}n_\zeta^{1-\delta_{U_d}^{\zeta}}Z^{(2)}_{(U_d;\ t_{u_1},\cdots,t_{u_d})}\frac{\alpha(u_1,\cdots,u_d;\ t_{u_1},\cdots,t_{u_d})}{G_{(U)}}$$

$$-\frac{1}{2}\prod_{j=0}^{r} n_j (Z^{(1)})^2 \frac{1}{G} - \frac{1}{2}\sum_{d=1}^{r-s} \sum_{U_d \subset R-S} Z^{(3)}_{(U_d)} \frac{1}{G_{(U_d)}}$$

$$-\frac{1}{2}\sum_{k=1}^{s} \sum_{I_k \subset S} \sum_{h=0}^{r-s} \sum_{J_h \subset R-S} S_{(I_k J_h)} \frac{1}{C_{(I_k)} D_{(J_h)}} - \frac{1}{2}\frac{S_0}{\sigma_0^2}\Bigg],$$

where

(4.94) $$Z^{(1)} = \bar{X}$$

(4.95) $$Z^{(2)}_{(U_d;\, t_{u_1}, \cdots, t_{u_d})} = \bar{X}_{t_{u_1}, \cdots, t_{u_d}} + \sum_{\gamma=1}^{d} \sum_{P_\gamma \subset U_d} (-1)^\gamma \bar{X}_{t_{p_1} \cdots t_{p_{d-\gamma}} n_{p_{d-\gamma+1}} \cdots n_{p_d}},$$
$(U_d \subset R-S;\ d=1, \cdots, r-s),$

(4.96) $$Z^{(3)}_{(U_d)} = \prod_{\zeta=0}^{r} n_\zeta^{1-\delta^{\zeta}_{U_d}} \sum_{t_{u_1}, \cdots, t_{u_d}} \left(\sum_{\beta=0}^{d} \sum_{L_\beta \subset U_d} (-1)^{d-\beta} \bar{X}_{t_{l_1} \cdots t_{l_\beta}} \right)^2,$$
$(U_d \subset R-S;\ d=1, \cdots, r-s).$

This lemma can be easily proved in virture of (3.53).

Now we shall consider about the functional relationship among (4.94), (4.95) and (4.96).

At first let us observe that $\bar{X}_{n_{u_1}}$ is determined uniquely by determining $Z^{(1)}$ and $\{Z^{(2)}_{(u_1;\, t_{u_1})};\ t_{u_1}=1, 2, \cdots, n_{u_1}-1\}$, which implies the unique determination of $Z^{(3)}_{(u_1)}$ and $\bar{X}_{n_{u_2}}$ is determined uniquely by determining $Z^{(1)}$ and $\{Z^{(2)}_{(u_2;\, t_{u_2})};\ t_{u_2}=1, 2, \cdots, n_{u_2}-1\}$, which implies again the unique determination of $Z^{(3)}_{(u_2)}$, and the analogue holds true for all sets $u_i \subset R-S$.

Secondly we observe that $\bar{X}_{n_{u_1} n_{u_2}}$ is determined uniquely by determining $Z_{(1)}$, $\bar{X}_{n_{u_1}}$, $\bar{X}_{n_{u_2}}$ and $\{Z^{(2)}_{(U_2;\, t_{u_1}, t_{u_2})};\ t_{u_1}=1, \cdots, n_{u_1}-1,\ t_{u_2}=1, \cdots, n_{u_2}-1\}$, which implies also the unique determination of $Z^{(3)}_{(u_1, u_2)}$, and the analogue of them holds true for all sets $(u_i, u_j) \subset R-S$.

In general it holds that $\bar{X}_{n_{j_1} n_{j_2} \cdots n_{j_h}}$ $(J_h \subset R-S)$ is determined uniquely by determining $Z^{(1)}$, $\{\bar{X}_{n_{u_1} n_{u_2} \cdots n_{u_d}};\ U_d \subset J_h,\ d=1, 2, \cdots, h-1\}$ and $\{Z^{(2)}_{(J_h;\, t_{j_1}, \cdots, t_{j_h})};\ t_{j_c}=1, \cdots, n_{j_c}-1,\ c=1, 2, \cdots, h\}$, which implies also the unique determination of $Z^{(3)}_{(J_h)}$. Thus it can be seen that $\{Z^{(3)}_{(U_d)};\ U_d \subset R-S,\ d=1, 2, \cdots, r-s\}$ are determined uniquely by determining $Z^{(1)}$ and $\{Z^{(2)}_{(U_d;\, t_{u_1}, \cdots, t_{u_d})};\ U_d \subset R-S,\ d=1, 2, \cdots, r-s,\ t_{u_j}=1, \cdots, n_{u_j}-1,\ j=1, \cdots, d\}$.

Now, in our mixed model of Type II, the random variable X and the sample space R^X are the same as those given in Type I, and the family $\mathfrak{P}^X$ is specified by the parameter $\theta = (\mu, \alpha(j_1, \cdots, j_h;\, t_{j_1}, \cdots, t_{j_h}),\ \sigma^2_{I_k J_h},\ \sigma_0^2;\ I_k \subset S,\ J_h \subset R-S,\ t_{j_c}=1, \cdots, n_{j_c}-1,\ c=1, \cdots, h,\ h=1, \cdots, r-s, k=1, \cdots, s)$, whose space is of $\left(2^r - 2^{r-s} + \prod_{i=s+1}^{r} n_i + 1\right)$-dimension, where $-\infty < \mu$

$<\infty$, $-\infty<\alpha(j_1,\cdots,j_h;\ t_{j_1},\cdots,t_{j_h})<\infty$, $0<\sigma^2_{I_kJ_h}<\infty$, $0<\sigma^2_0<\infty$.

We shall consider the transformation of the original parameter such that

(4.97) $$\tau^{(1)}=\prod_{j=0}^{r} n_j \frac{\mu}{G},$$

(4.98) $$\tau^{(2)}(U_d;\ t_{u_1},\cdots,\ t_{u_d})=\prod_{\zeta=0}^{r} n_\zeta^{1-\delta^\zeta_{U_d}} \frac{\alpha(u_1,\cdots,u_d;\ t_{u_1},\cdots,t_{u_d})}{G_{(U_d)}},\quad \begin{pmatrix} U_d\subset R-S;\ t_{u_c}=1,\cdots,n_{u_c}-1, \\ c=1,\cdots,d;\ d=1,\cdots,r-s, \end{pmatrix},$$

(4.99) $$\tau^{(3)}_{(I_kJ_h)}=-\frac{1}{2C_{(I_k)}D_{(J_h)}},\quad \begin{pmatrix} I_k\subset S;\ J_h\subset R-S, \\ k=1,\cdots,s;\ h=1,\cdots,r-s \end{pmatrix},$$

(4.100) $$\tau^{(4)}=-\frac{1}{2\sigma_0^2}.$$

After observing the independency of the class of parametric functions $\{C_{(I_k)}D_{(J_h)};\ I_k\subset S, J_h\subset R-S,\ k=1,\cdots,s,\ h=1,\cdots,r-s\}$, it is seen that the transformation (4.97), $\cdots$, (4.100) from θ to $\tau=(\tau^{(1)},\ \tau^{(2)}_{(U_d;t_{u_1},\cdots,t_{u_d})},\ \tau^{(3)}_{(I_kJ_h)},\ \tau^{(4)};\ I_k\subset S,\ J_h\subset R-S,\ U_d\subset R-S,\ t_{u_c}=1,\cdots,n_{u_c}-1,\ c=1,\cdots,d,\ d=1,\cdots,r-s,\ k=1,\cdots,s,\ h=1,\cdots,r-s)$ is one-to-one. There fore we can say that $\mathfrak{P}^X$ is specified by τ, where $-\infty<\tau^{(1)}<\infty$, $-\infty<\tau^{(2)}_{(U_d;t_{u_1},\cdots,t_{u_d})}<\infty$, $-\infty<\tau^{(4)}<\tau^{(3)}_{(M)}<\tau^{(3)}_{(N)}<0$ for any pair $(M,\ N)$ such that $M\supset N$, $M\subset R$, $N\subset R$. (We should notice that G and $G_{(U_d)}$ in (4.93) are the functions of $C_{(I_k)}D_{(J_h)}$: see Definition 4.1).

Then, under the new parameter τ, by using the above-mentioned result about $Z^{(3)}_{(U_d)}$ we obtain the probability density function of X as follows,

(4.101) $$K\varphi_{(\tau)}exp\Big[\tau^{(1)}Z^{(1)}+\sum_{d=1}^{r-s}\sum_{U_d\subset R-S}\sum_{t_{u_1}=1}^{n_{u_1}-1}\cdots\sum_{t_{u_d}=1}^{n_{u_d}-1}\tau^{(2)}_{(U_d;t_{u_1},\cdots,t_{u_d})}Z^{(2)}_{(U_d;\ t_{u_1},\cdots,t_{u_d})}$$
$$+\sum_{k=1}^{s}\sum_{I_k\subset S}\sum_{h=0}^{r-s}\sum_{J_h\subset R-S}\tau^{(3)}_{(I_kJ_h)}S(I_kJ_h)+\tau^{(4)}S_0$$
$$+\sum_{i=1}^{2^{r-s}}g_i(\tau^{(3)}_{(M)})\,h_i(Z^{(1)},Z^{(2)}_{(N)})\Big].$$

Hence the sufficient statistic for $\mathfrak{P}^X$ is $U=(Z^{(1)},\ Z^{(2)}_{(U_d;\ t_{u_1},\cdots,\ t_{u_d})},\ S_{(I_hJ_k)},\ S_0;\ I_k\subset S,\ J_h\subset R-S,\ U_d\subset R-S,\ k=1,2,\cdots,\ s,\ h=1,2,\cdots,\ r-s,\ t_{u_c}=1,2,\cdots,\ n_{u_c}-1,\ c=1,2,\cdots,\ d,\ d=1,2,\cdots,\ r-s)$. As for completeness the result reviewed in Section 3 cannot be applied directly to the family of our probability density (4.102). In order to show $\mathfrak{P}^U$ in this type be complete, we need to generalize a result due to Gautschi [3], which is given by Lemma 4.8.

Let $U^{(k)}$ be a k-dimensional Euclidean space with the point $u^{(k)}=(u_1,\ u_2,\cdots,\ u_k)$. We shall write the first j components as $u^{(j)}$ and the remaining components $u^{((j))}$ so that we write $u^{(k)}$ in the following different fashion $u^{(k)}=(u^{(j)},\ u^{((j))})=(u^{(k-1)},\ u_k)$ etc. And let $U^{(j)}$ be a j-dimensional Euclidean space with the point $u^{(j)}=(u_1,\ u_2,\cdots,\ u_j)$, U_j be the j-th component space of $U^{(k)}$.

In the following lemma, $T^{(k)}$ denotes a k dimensional Euclidean space with the

point $\tau^{(k)}=(\tau_1, \tau_2, \cdots, \tau_k)$, and the notations such as $\tau^{(j)}$, T_j etc. are to be understood in the way above stated. Further, L with a subscript or a superscript denotes the Lebesgue measure, where the subscript j indicates the space U_j or T_j on which the measure is taken, and the superscript (j) indicates the space $U^{(j)}$ or $T^{(j)}$ on which the measure is taken.

LEMMA 4.8. *Let*

$$\mathfrak{P}^{u(k)}=\{P_{\tau(k)}^{u(k)} \mid \tau^{(k)}\in\omega\},$$

where ω is a Borel set in an Euclidean space containing a non-degenerate k-dimensional interval, be the family of measures $P_{\tau(k)}^{u(k)}$ on the additive family of subsets in the space R^U of point U, having the density

$$p_{\tau(k)}(u^{(k)})=C(\tau^{(k)})\, h(u^{(k)})\, exp\left[\sum_{i=1}^{k} \tau_i u_i+g(\tau^{(s)}, u^{((s))})\right] \tag{4.102}$$

with respect to Lebesgue measure. Then $\mathfrak{P}^{u(k)}$ is strongly complete.

PROOF. We shall assume, without loss of generality, $p_{\tau(k)}(u^{(k)})$ be defined for all $\tau^{(k)}$ and $u^{(k)}$ in the k-dimensional Euclidean spaces $T^{(k)}$ and $U^{(k)}$ respectively such that if it is not originally defined it is assumed to be equal to zero. Let $f(u^{(k-1)}, u_k)$ be $\mathfrak{B}$-measurable and integrable with respect to the Lebesgue measure.

Suppose that

$$I\equiv\int_{U^{(k-1)}\times U_k} f(u^{(k-1)}, u_k)\, p_{\tau(k)}(u^{(k)})\, d(m^{(k-1)}\times m_k)=0 \tag{4.103}$$

$$(a.e.\ L^{(k-1)}\times L_k=L^{(k)}).$$

Let $N^{(1)}$ be the set of parameter points $(\tau^{(k-1)}, \tau_k)$ for which $I\neq 0$ and $N_{(k-1)}^{(1)}$ be the $\tau^{(k-1)}$-section of $N^{(1)}$. Then $L_k(N_{(k-1)})=0$ except for $\tau_{(k-1)}\in N_0^{(1)}$, where $L^{(k-1)}(N_0^{(1)})=0$.

Hence, by Fubini's theorem, we have

$$I=\int_{U_k} e^{\tau_k u_k}\left\{\int_{U^{(k-1)}} f(u^{(k-1)}, u_k)\, C(\tau^{(k)})\, h(u^{(k)}) \cdot exp\left[\sum_{i=1}^{k-1} \tau_i u_i+g(\tau^{(s)}, u^{((s))})\right] dm^{(k-1)}\right\} dm_k=0 \quad (a.e.\ L_k) \tag{4.104}$$

holds true for all $\tau^{(k-1)}\notin N_0^{(1)}$, where integral of inside exists except for $u_k\in M_k$ such that $L_k(M_k)=0$.

By the unicity of the Laplace transform, we obtain

$$\begin{aligned} I_1(\tau^{(k-1)}, u_k)&=\int_{U^{(k-1)}} f(u^{(k-1)}, u_k)\, h(u^{(k)})\, exp\left[\sum_{i=1}^{k-1} \tau_i u_i+g(\tau^{(s)}, u^{((s))})\right] dm^{(k-1)} \\ &=0 \qquad (a.e.\ L_k),\ \tau^{(k-1)}\notin N_0^{(1)}. \end{aligned} \tag{4.105}$$

If $S^{(1)}$ denotes the set of points $(\tau^{(k-1)}, u_k)$ for which $I_1(\tau^{(k-1)}, u_1)$ is neither exists nor equal to zero, almost every $\tau^{(k-1)}$-section of $S^{(1)}$ has L_k-measure zero. Hence $L^{(k)}(S^{(1)})$

$=0$. As this implies that almost every u_k-section of $S^{(1)}$ has $L^{(k-1)}$-measure zero, it follows that

(4.106) $$I_1(\tau^{(k-1)}, u_k)=0 \qquad (a.e.\ L^{(k-1)}),\ u_k \notin M_k.$$

(4.106) is written by

(4.107) $$I_1(\tau^{(k-1)}, u_k)$$
$$=\int_{U^{(k-2)}\times U_{k-1}} f(u^{(k-2)}, u_{k-1}, u_k)h(u^{(k)})\, exp\left[\sum_{i=1}^{k-1}\tau_i u_i+g(\tau^{(s)}, u^{(s)})\right] d(m^{(k-2)}\times m_{k-1})$$
$$=0 \qquad (a.e.\ L^{(k-2)}\times L_{k-1}=L^{(k-1)}),\ u_k \notin M_k.$$

Let $N^{(2)}$ be the set of parameter points $(\tau^{(k-2)}, \tau_{k-1})$ for which $I_1(\tau^{(k-1)}, u_k)\neq 0$, and $N^{(2)}_{(k-2)}$ be the $\tau^{(k-2)}$-section of $N^{(2)}$. Then $L_{k-1}(N^{(2)}_{(k-2)})=0$ except for $\tau^{(k-2)}\in N^{(2)}_0$, where $L^{(k-2)}(N^{(2)}_0)=0$.

Hence, again by Fubini's theorem, we have

(4.108) $$I_1(\tau^{(k-1)}, u_k)$$
$$=\int_{U_{k-1}} e^{\tau_{k-1}u_{k-1}}\left\{\int_{U^{(k-2)}} f(u^{(k-2)}, u_{k-1}, u_k)\, h(u^{(k)})\right.$$
$$\left.\cdot\ exp\left[\sum_{i=1}^{k-2}\tau_i u_i+g(\tau_{(s)}, u^{((s))})\right] dm^{(k-2)}\right\} dm_{k-1}$$
$$=0 \qquad (a.e.\ L_{k-1}),\ u_k \notin M_k$$

holds true for all $\tau^{(k-2)}\notin N^{(2)}_0$, where integral of inside exists except for $u_{k-1}\in M_{k-1}$ such that $L_{k-1}(M_{k-1})=0$.

By the unicity of the Laplace transform, we obtain

(4.109) $$I_2(\tau^{(k-2)}, u_{k-1}, u_k)$$
$$=\int_{U^{(k-2)}} f(u^{(k-2)}, u_{k-1}, u_k)\, h\,(u^{(k)})\, exp\left[\sum_{i=1}^{k-2}\tau_i u_i+g(\tau^{(s)}, u^{((s))})\right] dm^{(k-2)}$$
$$=0 \qquad (a.e.\ L_{k-1}),\ \tau^{(k-2)}\notin N^{(2)}_0,\ u_k\notin M_k.$$

If $S^{(2)}_{(u_k)}$ denotes the set of points $(\tau^{(k-2)}, u_{k-1})$ for which $I_2(\tau^{(k-2)}, u_{k-1}, u_k)$, corresponding to some fixed $u_k\notin M_k$, neither exists nor is equal to zero, almost every $\tau^{(k-2)}$-section of $S^{(2)}_{(u_k)}$ has L_{k-1}-measure zero.

Hence we have

$$L^{(k-1)}(S^{(2)}_{(u_k)})=0, \qquad u_k\notin M_k.$$

And we obtain

(4.110) $$I_2(\tau^{(k-2)}, u_{k-1}, u_k)=0 \qquad (a.e.\ L^{(k-2)}),\ u_{k-1}\notin M_{k-1},\ u_k\notin M_k.$$

In this way we can proceed inductively and obtain finally

$$(4.111) \qquad I_{k-s}(\tau^{(s)}, u^{((s))})$$

$$= e^{g(\tau^{(s)}, u^{((s))})} \int_{U(s)} f(u^{(s)}, u^{((s))})\, h(u^{(s)}, u^{((s))}) \ exp\left[\sum_{i=1}^{s} \tau_i u_i\right] dm^{(s)}$$

$$= 0 \qquad (a.e.\ L^{(s)}),\ u_{s+1} \notin M_{s+1},\ u_{s+2} \notin M_{s+2}, \cdots,\ u_k \notin M_k,$$

where $\quad L^j(M_j) = 0 \qquad (j = s+1, \cdots, k).$

On the other hand, as was mentioned in Section 3, the family of measures whose density is given in the form $h'(u^{(s)})\ exp\left[\sum_{i=1}^{s} \tau_i u_i\right]$ is strongly complete. Therefore (4.111) implies that

$$(4.112) \qquad f(u^{(s)}, u^{((s))}) = 0 \qquad (a.e.\mathfrak{B}^{u^{(s)}}),\ u_j \notin M_j\ (j = s+1, \cdots, k),$$

which again implies

$$(4.113) \qquad f(u^{(s+1)}, u^{((s+1))}) = 0 \qquad (a.e.\ \mathfrak{B}^{u^{(s+1)}}),\ u_j \notin M_j\ (j = s+2, \cdots, k).$$

In this way we finally obtain

$$(4.114) \qquad f(u^{(k)}) = 0 \qquad (a.e.\ \mathfrak{B}^{u^{(k)}}),$$

which completes the proof.

Now, as the probability density function of the sufficient statistic U for $\mathfrak{B}^X$ in our case is written in the form (4.102) in Lemma 4.8, also in the case of Type II we have the same conclusion about the estimation problem for the unknown parameters as we obtained in Theorem 3.4 in Section 3.

5. Remark on the random effect model.

At first it should be remarked here that the result of the previous paper of the author concerning the point estimation of the variance components in random effect model can be improved to the level of the generality of this paper by applying the Lemma 4.8 of this paper. The author would like to mention the work of L. H. Herbach [5] concerning the problem of testing in the same model, which seems to be restricted to the case of one and two way layout only and about the testing of variance components belonging to a certain group can be generalized to the case of multi-way layout by making use of the joint density function derived in the previous paper of the author, while there exists no exact F-test of them belonging to another group.

6. Acknowledgement.

The author would like to express his hearty thanks to Professor T. Kitagawa and Dr. A. Kudô of the mathematical institute, Kyushu University for their encouragements and criticisms while this paper was being prepared.

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ON THE COHOMOLOGY SPACE OF LIE TRIPLE SYSTEM

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(Received September 30, 1960)

It is well known that a Lie group can be characterized locally by a Lie algebra. More generally, the algebraic system which characterizes locally a totally geodesic subspace in a group space or a symmetric space is a Lie triple system [1, 9, 10][1]. A Lie algebra and a special Jordan algebra are the typical examples which may be a Lie triple system and the systematic study of this system was done by N. Jacobson [6] and W. G. Lister [7]. In this paper, we give a method defining a cohomology space of a Lie triple system and a relation between a cohomology space of order 3 and an extension of a Lie triple system. Next, we prove for a non-degenerate Lie triple system an analogue of the Casimir theorem. We see that an identity (3), which is called the Ricci formula in the differential geometry, plays a fundamental role in this study.

Recently, Professor B. Harris studied on the cohomology of Lie triple system independently of us [3], his detailed results will appear in [4]. The author wishes to express his sincere thanks for his kind informations.

1. We begin with the definition of the Lie triple system.

DEFINITION 1. A *Lie triple system* (L. t. s.) is a vector space $\mathfrak{T}$ over a field Φ[2], which is closed with respect to a trilinear multiplication $[abc]$ and satisfying

$$(1)\qquad [aab]=0,$$

$$(2)\qquad [abc]+[bca]+[cab]=0,$$

$$(3)\qquad [[abc]de]+[[bad]ce]+[ba[cde]]+[cd[abe]]=0.$$

PROPOSITION 1.[3] In L. t. s. it holds the following identities:

$$(4)\qquad [[abc]de]+[[bad]ce]+[[cda]be]+[[dcb]ae]=0,$$

$$\begin{aligned}(5)\qquad &[[[abc]de]fg]+[[[bac]df]eg]+[[[abd]cf]eg]+[[[bad]ce]fg]\\ &+[[[cde]fa]bg]+[[[dce]fb]ag]+[[[cdf]eb]ag]+[[[dcf]ea]bg]\\ &+[[[efa]bc]dg]+[[[fea]bd]cg]+[[[efb]ad]cg]+[[[feb]ac]dg]=0.\end{aligned}$$

PROOF. Interchanging pairs (a, b) and (c, d) in (3), we have

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1) Numbers in brackets refer to the references at the end of the paper.

2) Throughout this paper we shall assume that the characteristic of the base field Φ is 0 and L. t. s. has a finite dimension. See [6, 7, 10] as to the terminologies for L. t. s. in this paper.

3) These identities were first stated by N. Jacobson [5] and W. G. Lister first pointed out that (1), (2), (3) imply (4), (5), but he did not publish. This is derived also from [5, § 3] and [10, Theorem 2. 1].

$$(3)' \quad [[cda]be]+[[dcb]ae]+[dc[abe]]+[ab[cde]]=0.$$

The addition of (3) and (3)$'$ implies (4). For a proof of (5) we use twice (3).

$$\begin{aligned}&\mathfrak{S}\{[[[abc]de]fg]+[[[bac]df]eg]+[[[abd]cf]eg]+[[[bad]ce]fg]\}\\ =&\mathfrak{S}\{[[abc]d[efg]]-[ef[[abc]dg]]+[[abd]c[feg]]-[fe[[abd]cg]]\}\\ =&\mathfrak{S}\{[ab[cd[efg]]]-[cd[ab[efg]]]-[ef[ab[cdg]]]+[ef[cd[abg]]]\}\\ =&0,\end{aligned}$$

where $\mathfrak{S}$ denotes the summation obtained by cyclic permutations of the pairs (a, b), (c, d), (e, f).

DEFINITION 2. Let $\mathfrak{T}$ be a L. t. s. and let V be a vector space over Φ. Suppose that there exists a bilinear mapping θ: $(a, b)\rightarrow\theta(a, b)$ of $\mathfrak{T}\times\mathfrak{T}$ into an associative algebra of linear transformations of V. Then, V is called a $\mathfrak{T}$ *module* if θ satisfies the following conditions:

$$(6) \qquad \theta(c, d)\theta(a, b)-\theta(b, d)\theta(a, c)-\theta(a, [bcd])+D(b, c)\theta(a, d)=0,$$

$$(7) \qquad \theta(c, d)D(a, b)-D(a, b)\theta(c, d)+\theta([abc], d)+\theta(c, [abd])=0,$$

where

$$(8) \qquad D(a, b)=\theta(b, a)-\theta(a, b).$$

From (7) we obtain

$$(9) \qquad D(c, d)D(a, b)-D(a, b)D(c, d)+D([abc], d)+D(c, [abd])=0,$$

hence the vector space spanned by $\sum_i D(a_i, b_i)$, $a_i, b_i\in\mathfrak{T}$ is a subalgebra of $\mathfrak{Gl}(V)$.

In a L. t. s. $\mathfrak{T}$, let $\theta(a, b)$ be a linear mapping $x\rightarrow[xab]$ of $\mathfrak{T}$ into itself, a, b being in $\mathfrak{T}$, then we can prove that $\mathfrak{T}$ is a $\mathfrak{T}$-module by using (3), and in this case $D(a, b)$ becomes a linear mapping $x\rightarrow[abx]$ by (2) (inner derivation). An ideal of L. t. s. $\mathfrak{T}$ is an invariant subspace of the mappings $\theta(a, b)$ for all a, b in $\mathfrak{T}$.

REMARK. Let $a\rightarrow R_a$ be a linear mapping of L. t. s. $\mathfrak{T}$ into an associative algebra of linear transformations of a vecter space V and satisfies $R_{[abb]}=[[R_a R_b] R_b]$ for all a, b in $\mathfrak{T}$, where $[R_a R_b]\equiv R_aR_b-R_bR_a$. Then, from $R_{[a\, b+c\, b+c]}=[[R_a\, R_b+R_c]\, R_b+R_c]$, it follows $R_{[abc]}+R_{[acb]}=[[R_a R_b] R_c]+[[R_a R_c] R_b]$. Hence $R_{[bac]}+R_{[bca]}=[[R_b R_a] R_c]+[[R_b R_c] R_a]$. By using (2) and the Jacobi identity, from the last two relations we have $R_{[abc]}=[[R_a R_b] R_c]$. If we put $\theta(a, b)=R_bR_a$, then $D(a, b)=[R_a R_b]$. Since these operators satisfy (6) and (7), it follows that V is a $\mathfrak{T}$-module.

W. G. Lister [7] defined a representation of L. t. s. in a natural sense as a L. t. s. homomorphism of a L. t. s. into a L. t. s. of linear transformations of a vector space. Therefore, the mapping θ may be considered as a representation of L. t. s. in a general sense.

Let V be a $\mathfrak{T}$-module defined by a bilinear mapping θ and let f be an n-linear mapping of $\underbrace{\mathfrak{T}\times\cdots\times\mathfrak{T}}_{n\ \text{times}}$ into V satisfying

$$f(x_1, x_2, \cdots, x_{n-3}, x, x, x_n)=0$$

and

$$f(x_1, x_2, \cdots, x_{n-3}, x, y, z)+f(x_1, x_2, \cdots, x_{n-3}, y, z, x)+f(x_1, x_2, \cdots, x_{n-3}, z, x, y)=0.$$

We denote the vector space spanned by such n-linear mappings by $C^n(\mathfrak{T}, V)$, $(n=0, 1, 2, \cdots)$, where we define $C^0(\mathfrak{T}, V)=V$.

Next, we define a linear mapping δ of $C^n(\mathfrak{T}, V)$ into $C^{n+2}(\mathfrak{T}, V)$ by the following formulas:

$$\delta f(x_1, x_2)=\theta(x_1, x_2)f \qquad \text{for } f\in C^0(\mathfrak{T}, V), \tag{10}$$

$$\begin{aligned}
&\delta f(x_1, x_2, \cdots, x_{2n+1})\\
&=\theta(x_{2n}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-1})-\theta(x_{2n-1}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-2}, x_{2n})\\
&\quad+\sum_{k=1}^{n}(-1)^{n+k}D(x_{2k-1}, x_{2k})f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1})\\
&\quad+\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{n+k+1}f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2n+1})\\
&\qquad\qquad\text{for } f\in C^{2n-1}(\mathfrak{T}, V), n=1, 2, 3, \cdots,
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\delta f(y, x_1, x_2, \cdots, x_{2n+1})\\
&=\theta(x_{2n}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-1})-\theta(x_{2n-1}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-2}, x_{2n})\\
&\quad+\sum_{k=1}^{n}(-1)^{n+k}D(x_{2k-1}, x_{2k})f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1})\\
&\quad+\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{n+k+1}f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2n+1})\\
&\qquad\qquad\text{for } f\in C^{2n}(\mathfrak{T}, V), n=1, 2, 3, \cdots,
\end{aligned} \tag{12}$$

where the sign ^ over a letter indicates that this letter is to be omitted. The operator δ is as follows for lower orders:

If $n=1$, $\delta f(x_1, x_2, x_3)=\theta(x_2, x_3)f(x_1)-\theta(x_1, x_3)f(x_2)+D(x_1, x_2)f(x_3)-f([x_1x_2x_3])$,

$n=2$, $\delta f(x_1, x_2, x_3, x_4)=\theta(x_3, x_4)f(x_1, x_2)-\theta(x_2, x_4)f(x_1, x_3)+D(x_2, x_3)f(x_1, x_4) -f(x_1, [x_2x_3x_4])$.

Now, if $f\in C^0(\mathfrak{T}, V)$, then

$$\begin{aligned}
&\delta\delta f(x_1, x_2, x_3, x_4)\\
&=\theta(x_3, x_4)\delta f(x_1, x_2)-\theta(x_2, x_4)\delta f(x_1, x_3)+D(x_2, x_3)\delta f(x_1, x_4)-\delta f(x_1, [x_2x_3x_4])\\
&=(\theta(x_3, x_4)\theta(x_1, x_2)-\theta(x_2, x_4)\theta(x_1, x_3)+D(x_2, x_3)\theta(x_1, x_4)-\theta(x_1, [x_2x_3x_4]))f\\
&=0
\end{aligned}$$

by (6). Similarly, $\delta\delta f=0$ for $f\in C^1(\mathfrak{T}, V)$ by (3), (6), (7), (9).

For $a, b\in\mathfrak{T}$ we define a linear mapping $\kappa(a, b)$ of $C^{2n-1}(\mathfrak{T}, V)$ into $C^{2n-1}(\mathfrak{T}, V)$ and a linear mapping $\iota(a, b)$ of $C^{2n-1}(\mathfrak{T}, V)$ into $C^{2n-3}(\mathfrak{T}, V)$ as follows:

$$(13)\qquad (\kappa(a, b)f)(x_1, \cdots, x_{2n-1})$$
$$=(-1)^{n+1}\left(D(a, b)f(x_1, \cdots, x_{2n-1})-\sum_{j=1}^{2n-1}f(x_1, \cdots, [abx_j], \cdots, x_{2n-1})\right),$$

$$(14)\qquad (\iota(a, b)f)(x_1, \cdots, x_{2n-3})=f(a, b, x_1, \cdots, x_{2n-3}),$$

$$n=2, 3, \cdots$$

Then we have the following relations.

LEMMA 1. *For $a, b, c, d\in\mathfrak{T}$ and $f\in C^{2n-1}(\mathfrak{T}, V)$ $(n=2, 3, \cdots)$*

(i) $$(\iota(a, b)\delta-\delta\iota(a, b))f=\kappa(a, b)f,$$

(ii) $$(\kappa(a, b)\iota(c, d)+\iota(c, d)\kappa(a, b))f=(-1)^n(\iota([abc], d)+\iota(c, [abd]))f,$$

(iii) $$(\kappa(a, b)\kappa(c, d)-\kappa(c, d)\kappa(a, b))f=(-1)^{n+1}(\kappa([abc], d)+\kappa(c, [abd]))f,$$

(iv) $$(\delta\kappa(a, b)+\kappa(a, b)\delta)f=0.$$

PROOF. Since it is easy to prove (i) and (ii), we shall prove (iii) and (iv).

(iii): If $f\in C^3(\mathfrak{T}, V)$, then it follows easily (iii). Hence, we assume (iii) holds for $f\in C^{2n-3}(\mathfrak{T}, V)$. Then for $f\in C^{2n-1}(\mathfrak{T}, V)$ and arbitrary $k, l\in\mathfrak{T}$

$$\begin{aligned}
&\iota(k, l)(\kappa(a, b)\kappa(c, d)-\kappa(c, d)\kappa(a, b)+(-1)^n\kappa([abc], d)+(-1)^n\kappa(c, [abd]))f\\
&=-\kappa(a, b)\iota(k, l)\kappa(c, d)f+(-1)^n\iota([abk], l)\kappa(c, d)f+(-1)^n\iota(k, [abl])\kappa(c, d)f\\
&\quad+\kappa(c, d)\iota(k, l)\kappa(a, b)f-(-1)^n\iota([cdk], l)\kappa(a, b)f-(-1)^n\iota(k, [cdl])\kappa(a, b)f\\
&\quad+(-1)^n\iota(k, l)\kappa([abc], d)f+(-1)^n\iota(k, l)\kappa(c, [abd])f\\
&=(\kappa(a, b)\kappa(c, d)-\kappa(c, d)\kappa(a, b)-(-1)^n\kappa([abc], d)-(-1)^n\kappa(c, [abd]))\iota(k, l)f\\
&\quad+(\iota([cd[abk]], l)-\iota([ab[cdk]], l)+\iota([[abc]dk], l)-\iota([[abd]ck], l))f\\
&\quad+(\iota(k, [cd[abl]])-\iota(k, [ab[cdl]])+\iota(k, [[abc]dl])-\iota(k, [[abd]cl]))f\\
&=0,
\end{aligned}$$

by (ii) and (3). Therefore, (iii) holds for $f\in C^{2n-1}(\mathfrak{T}, V)$.

(iv): For $f\in C^3(\mathfrak{T}, V)$ we obtain (iv). Therefore, we assume that (iv) holds for all $f\in C^{2n-3}(\mathfrak{T}, V)$. Then, in the case $f\in C^{2n-1}(\mathfrak{T}, V)$, by using (i), (ii), (iii) for arbitrary $c, d\in\mathfrak{T}$

$$\begin{aligned}
&\iota(c, d)\delta\kappa(a, b)f+\iota(c, d)\kappa(a, b)\delta f\\
&=\delta\iota(c, d)\kappa(a, b)f+\kappa(c, d)\kappa(a, b)f-\kappa(a, b)\iota(c, d)\delta f\\
&\qquad\qquad+(-1)^{n+1}\iota([abc], d)\delta f+(-1)^{n+1}\iota(c, [abd])\delta f\\
&=-(\delta\kappa(a, b)+\kappa(a, b)\delta)\iota(c, d)f\\
&\qquad-(\kappa(a, b)\kappa(c, d)-\kappa(c, d)\kappa(a, b)+(-1)^n\kappa([abc], d)+(-1)^n\kappa(c, [abd]))f\\
&=0,
\end{aligned}$$

by the inductive assumption. Hence, (iv) holds for $f\in C^{2n-1}(\mathfrak{T}, V)$.

For every $a, b\in\mathfrak{T}$ and $f\in C^{2n-1}(\mathfrak{T}, V)$ $(n=2, 3, \cdots)$, by using Lemma 1 and the induction we obtain

$$\begin{aligned}\iota(a, b)(\delta\delta f)&=\delta\iota(a, b)\delta f+\kappa(a, b)\delta f\\&=\delta\delta\iota(a, b)f+\delta\kappa(a, b)f+\kappa(a, b)\delta f\\&=0,\end{aligned}$$

hence $\delta\delta f=0$ for $f\in C^{2n-1}(\mathfrak{T}, V)$ $(n=1, 2, \cdots)$. Then it follows immediately that $\delta\delta f=0$ for $f\in C^{2n}(\mathfrak{T}, V)$ $(n=1, 2, \cdots)$.

Thus we have the following main theorem.

THEOREM 1. *For the operator δ defined above, it holds that $\delta\delta f=0$ for any $f\in C^n(\mathfrak{T}, V)$, $n=0, 1, 2, \cdots$.*

The mapping $f\in C^n(\mathfrak{T}, V)$ is called a *cocycle* of order n if $\delta f=0$. We denote by $Z^n(\mathfrak{T}, V)$ a subspace spanned by cocycles of order n. The element of $B^n(\mathfrak{T}, V)=\delta C^{n-2}(\mathfrak{T}, V)$ is a *coboundary*. From Theorem 1, $B^n(\mathfrak{T}, V)$ is a subspace of $Z^n(\mathfrak{T}, V)$. Therefore we can define a *cohomology space* $H^n(\mathfrak{T}, V)$ of order n of $\mathfrak{T}$ as the factor space $Z^n(\mathfrak{T}, V)/B^n(\mathfrak{T}, V)$, $(n=0, 1, 2, \cdots)$.

2.[4] DEFINITION 3. Let $\mathfrak{T}, \mathfrak{U}, \mathfrak{m}$ be L.t.s. over the same base field. $\mathfrak{T}$ is an *extension of* $\mathfrak{U}$ *by* $\mathfrak{m}$ if there exists an exact sequence of L. t. s.:

$$0\longrightarrow\mathfrak{m}\xrightarrow{\ \iota\ }\mathfrak{T}\xrightarrow{\ \pi\ }\mathfrak{U}\longrightarrow 0.$$

Two extensions $\mathfrak{T}$ and $\mathfrak{T}'$ are said to be *equivalent* if the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m} \arrow[r] \arrow[d, "1"] & \mathfrak{T} \arrow[r] \arrow[d] & \mathfrak{U} \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & \mathfrak{m} \arrow[r] & \mathfrak{T}' \arrow[r] & \mathfrak{U} \arrow[r] & 0 \end{tikzcd}\ .$$

As a special case of a solvable ideal in a L. t. s. defined by W. G. Lister [7], we define an abelian ideal $\mathfrak{m}$ in $\mathfrak{T}$ as an ideal such that $[\mathfrak{T}\mathfrak{m}\mathfrak{m}]=(0)$. We consider the case that $\mathfrak{T}$ is an extension of $\mathfrak{U}$ by abelian ideal $\mathfrak{m}$ in $\mathfrak{T}$, that is $[\mathfrak{T}\iota(\mathfrak{m})\iota(\mathfrak{m})]=(0)$. Then, for elements $u=x+p, v=y+q$ $(x, y\in\mathfrak{T}, p, q\in\mathfrak{m})$, $\theta(u, v)m=[muv]=[mxy]$. Therefore, $\mathfrak{m}$ is an $\mathfrak{U}$-module by defining

$$\theta(u, v)m=[mtt'] \qquad \text{for any } t, t' \text{ in } \mathfrak{T} \text{ such that } \pi(t)=u, \pi(t')=v.$$

Let l be a section of the extension $\mathfrak{T}$ of $\mathfrak{U}$ by an abelian ideal in $\mathfrak{T}$, that is, l is a linear mapping of $\mathfrak{U}$ into $\mathfrak{T}$ such that $\pi l=1$. Next, we put

$$f(x_1, x_2, x_3)=[l(x_1)l(x_2)l(x_3)]-l([x_1x_2x_3]) \qquad x_i\in\mathfrak{U}\ (i=1, 2, 3), \tag{15}$$

4) In this section, we follow the method in [2].

then, f is a trilinear mapping of $\mathfrak{U}\times\mathfrak{U}\times\mathfrak{U}$ into $\mathfrak{m}$, since π is a homomorphism of $\mathfrak{T}$ onto $\mathfrak{U}$, and f belongs to $C^3(\mathfrak{U}, \mathfrak{m})$. We identify $\mathfrak{m}\times\mathfrak{U}$ and $\mathfrak{T}$ as vector spaces by $(m, x)\to m+l(x)$. In $\mathfrak{T}$ the following relation holds:

$$
\begin{aligned}
&[m_1+l(x_1)\ m_2+l(x_2)\ m_3+l(x_3)]\\
&\quad=[m_1l(x_2)l(x_3)]-[m_2l(x_1)l(x_3)]+[l(x_1)l(x_2)m_3]+f(x_1, x_2, x_3)+l([x_1x_2x_3]).
\end{aligned}
$$

Hence we can define a Lie triple product on $\mathfrak{m}\times\mathfrak{U}$ by

$$
\begin{aligned}
(16)\qquad &[(m_1, x_1)(m_2, x_2)(m_3, x_3)]\\
&\quad=(\theta(x_2, x_3)m_1-\theta(x_1, x_3)m_2+D(x_1, x_2)m_3+f(x_1, x_2, x_3), [x_1x_2x_3]).
\end{aligned}
$$

From this we obtain

$$[(m_1, x_1)(m_1, x_1)(m_2, x_2)]=(f(x_1, x_1, x_2), 0),$$

$$
\begin{aligned}
&[(m_1, x_1)(m_2, x_2)(m_3, x_3)]+[(m_2, x_2)(m_3, x_3)(m_1, x_1)]+[(m_3, x_3)(m_1, x_1)(m_2, x_2)]\\
&\quad=(f(x_1, x_2, x_3)+f(x_2, x_3, x_1)+f(x_3, x_1, x_2), 0),
\end{aligned}
$$

$$
\begin{aligned}
&[[(m_1, x_1)(m_2, x_2)(m_3, x_3)](m_4, x_4)(m_5, x_5)]\\
&+[[(m_2, x_2)(m_1, x_1)(m_4, x_4)](m_3, x_3)(m_5, x_5)]\\
&+[(m_2, x_2)(m_1, x_1)[(m_3, x_3)(m_4, x_4)(m_5, x_5)]]\\
&+[(m_3, x_3)(m_4, x_4)[(m_1, x_1)(m_2, x_2)(m_5, x_5)]]
\end{aligned}
$$

$$
=\left(\begin{aligned}
&(\theta(x_4, x_5)\theta(x_2, x_3)-\theta(x_3, x_5)\theta(x_2, x_4)-\theta(x_2, [x_3x_4x_5])+D(x_3, x_4)\theta(x_2, x_5))m_1\\
&-(\theta(x_4, x_5)\theta(x_1, x_3)-\theta(x_3, x_5)\theta(x_1, x_4)-\theta(x_1, [x_3x_4x_5])+D(x_3, x_4)\theta(x_1, x_5))m_2\\
&+(\theta(x_4, x_5)D(x_1, x_2)+D(x_2, x_1)\theta(x_4, x_5)-\theta([x_2x_1x_4], x_5)+\theta(x_4, [x_1x_2x_5]))m_3\\
&+(\theta(x_3, x_5)D(x_2, x_1)-D(x_2, x_1)\theta(x_3, x_5)-\theta([x_2x_1x_3], x_5)-\theta(x_3, [x_2x_1x_5]))m_4\\
&+(D(x_3, x_4)D(x_1, x_2)+D(x_2, x_1)D(x_3, x_4)-D([x_1x_2x_3], x_4)+D([x_1x_2x_4], x_3))m_5\\
&+\theta(x_4, x_5)f(x_1, x_2, x_3)-\theta(x_3, x_5)f(x_1, x_2, x_4)-D(x_1, x_2)f(x_3, x_4, x_5)\\
&+D(x_3, x_4)f(x_1, x_2, x_5)+f([x_1x_2x_3], x_4, x_5)+f(x_3, [x_1x_2x_4], x_5)\\
&+f(x_3, x_4, [x_1x_2x_5])-f(x_1, x_2, [x_3x_4x_5])
\end{aligned}, 0\right)
$$

$$=(\delta f(x_1, x_2, x_3, x_4, x_5), 0),$$

in which we used (3), (6), (7), (9). Therefore f is a cocycle of order 3.

Conversely, let $\mathfrak{m}$ be a $(\mathfrak{U}, \theta)$-module and abelian L. t. s. and let f be a cocycle of order 3. We define a ternary product on a vector space $\mathfrak{m}\times\mathfrak{U}$ by (16), then the vector space $\mathfrak{T}=\mathfrak{m}\times\mathfrak{U}$ becomes a L. t. s. with respect to this composition. Next we define the exact sequence:

$$0\longrightarrow\mathfrak{m}\xrightarrow{\ \iota\ }\mathfrak{T}\xrightarrow{\ \pi\ }\mathfrak{U}\longrightarrow 0$$

by $\iota(m)=(m, 0)$ and $\pi(m, x)=x$. Since ι and π are homomorphism, $\mathfrak{T}$ is an extension of $\mathfrak{U}$ by $\mathfrak{m}$, and it is easy to see that $\iota(\mathfrak{m})$ is an abelian in $\mathfrak{T}$. For a special section l: $l(x)=(0, x)$ $(x\in\mathfrak{U})$

$$[l(x_1)l(x_2)l(x_3)]-l([x_1x_2x_3])=(f(x_1, x_2, x_3), 0),$$

hence f is a cocycle defined by this extension.

If there exists another section l', then $g(x)=l'(x)-l(x)$ in $\mathfrak{m}$, and $f'(x_1, x_2, x_3) =[l'(x_1)l'(x_2)l'(x_3)]-l'([x_1x_2x_3])=f(x_1, x_2, x_3)+\theta(x_2, x_3)g(x_1)-\theta(x_1, x_3)g(x_2)+D(x_1, x_2)g(x_3)-g([x_1x_2x_3])=f(x_1, x_2, x_3)+\delta g(x_1, x_2, x_3)$, therefore f and f' belong to the same cohomology class.

Summarizing above results we have the following

THEOREM 2. *An extension $\mathfrak{T}$ of $\mathfrak{U}$ by an abelian ideal $\mathfrak{m}$ in $\mathfrak{T}$ defines an element of $H^3(\mathfrak{U}, \mathfrak{m})$. Conversely, if $\mathfrak{m}$ is abelian, an extension $\mathfrak{T}$ of $\mathfrak{U}$ by $\mathfrak{m}$ corresponds to any element of $H^3(\mathfrak{U}, \mathfrak{m})$ and $\mathfrak{m}$ becomes abelian in $\mathfrak{T}$.*

The extension: $0\longrightarrow\mathfrak{m}\xrightarrow{\iota}\mathfrak{T}\xrightarrow{\pi}\mathfrak{U}\longrightarrow 0$ is said to be inessential if there exists a subsystem $\mathfrak{T}'$ such that $\mathfrak{T}$ is a vector direct sum of $\iota(\mathfrak{m})$ and $\mathfrak{T}'$. Then

COROLLARY. *An extension $\mathfrak{T}$ of $\mathfrak{U}$ by an abelian ideal $\mathfrak{m}$ in $\mathfrak{T}$ is inessential if and only if $H^3(\mathfrak{U}, \mathfrak{m})=(0)$.*

3.[5] Let $\Theta(a, b)$ be a linear mapping $x\rightarrow[xba]$ of L. t. s. $\mathfrak{T}$, $a, b\in\mathfrak{T}$. Put $\phi(a, b) =Tr\Theta(a, b)$, and call this form ϕ a Killing form of $\mathfrak{T}$.

An 1-to-1 mapping A of $\mathfrak{T}$ is called an automorphism of $\mathfrak{T}$ if $A[xyz]=[AxAyAz]$ for all $x, y, z\in\mathfrak{T}$. A derivation D of $\mathfrak{T}$ is a linear mapping of $\mathfrak{T}$ such that $D[xyz] =[(Dx)yz]+[x(Dy)z]+[xy(Dz)]$ for all $x, y, z\in\mathfrak{T}$.

LEMMA 2. *Let $\phi(a, b)$ be a Killing form of L. t. s. $\mathfrak{T}$, then*

(i) $\phi(Ax, Ay)=\phi(x, y)$ *for automorphism A of $\mathfrak{T}$,*

(ii) $\phi(Dx, y)+\phi(x, Dy)=0$ *for derivation D of $\mathfrak{T}$,*

i. e. ϕ is an invariant of D.

PROOF. From the definition of an automorphism, we have $A\Theta(x, y)=\Theta(Ax, Ay)A$, hence $\phi(Ax, Ay)=Tr\Theta(Ax, Ay)=Tr(A\Theta(x, y)A^{-1})=Tr\Theta(x, y)=\phi(x, y)$. The proof is similar for the derivation, therefore we shall omit it.

Since the mapping: $x\rightarrow\sum_i[a_ib_ix]$ is an (inner) derivation of $\mathfrak{T}$, we have the following

COROLLARY. *A Killing form is an invariant of an inner derivation.*

Let (V, θ) be a $\mathfrak{T}$-module for L. t. s. $\mathfrak{T}$ and let $X_1, X_2, \cdots, X_n$ be a base of $\mathfrak{T}$. We call $\mathfrak{T}$ *non-degenerate* if

$$\det\begin{vmatrix}\phi(X_1, X_1), \cdots, \phi(X_1, X_n)\\ \vdots \qquad \vdots\\ \phi(X_n, X_1), \cdots, \phi(X_n, X_n)\end{vmatrix}\neq 0.$$

5) In this section, we follow the method in [8].

Then, we may define a linear operator C of V as

$$C=\sum_{i,j=1}^{n}\pi_{ji}\Theta(X_i, X_j),$$

where (π_{ij}) is an inverse matrix of $(\phi(X_i, X_j))$ and $\Theta(X_i, X_j)=\theta(X_j, X_i)$. We call this operator C a *Casimir operator* of θ. If we put $Y_i=\sum_{j=1}^{n}\pi_{ji}X_j$ $(i=1, 2, \cdots, n)$, $(Y_1, Y_2, \cdots, Y_n)$ is a base of $\mathfrak{T}$ and $\phi(X_i, Y_k)=\delta_{ik}$, and $C=\sum_{i=1}^{n}\Theta(X_i, Y_i)$.

Let $(X_1, \cdots, X_n)$ and $(X'_1, \cdots, X'_n)$ be bases of $\mathfrak{T}$ and let $\phi(X_i, X_j)$ and $\phi(X'_i, X'_j)$ be Killing forms with inverse matrix (π_{ij}) and (π'_{ij}) respectively. Denote by C and C' Casimir operators corresponding to bases (X_i) and (X'_i) respectively. Then, putting $X'_i=\sum_{j=1}^{n}a_{ij}X_j$, $Y'_i=\sum_{j=1}^{n}b_{ij}Y_j$, $C'=\sum_{i=1}^{n}\Theta(X'_i, Y'_i)=\sum_{i,s,t}a_{is}b_{it}\Theta(X_s, Y_t)=\sum_{s}\Theta(X_s, Y_s)=C$, since $\sum_{s}a_{is}b_{ks}=\sum_{s,t}a_{is}b_{kt}\delta_{st}=\sum_{s,t}a_{is}b_{kt}\phi(X_s, Y_t)=\phi(X'_i, Y'_k)=\delta_{ik}$. Hence, the Casimir operator is independent to the base of $\mathfrak{T}$.

THEOREM 3. *Let (V, θ) be a $\mathfrak{T}$-module of a non-degenerate L. t. s. $\mathfrak{T}$. Then the Casimir operator C of θ commutes with $D(x, y)$ for all x, y in $\mathfrak{T}$, where $D(x, y)=\theta(y, x)$ $-\theta(x, y)$.*

PROOF. From the fact that V is a $\mathfrak{T}$-module using (7) we have

$$\begin{aligned}D(x,y)C-CD(x,y)&=\sum_{i}\{D(x,y)\Theta(X_i, Y_i)-\Theta(X_i, Y_i)D(x,y)\}\\&=\sum_{i}\{\Theta([xyX_i], Y_i)+\Theta(X_i, [xyY_i])\},\end{aligned}$$

where $\Theta(X_i, Y_i)=\theta(Y_i, X_i)$. Putting $[xyX_i]=\sum_{j}a_{ij}X_j$ and $[xyY_i]=\sum_{j}b_{ij}Y_j$, it follows that $a_{ij}+b_{ji}=0$, because a Killing form is an invariant of an inner derivation. Hence

$$D(x,y)C-CD(x,y)=\sum_{i,k}\{a_{ik}\Theta(X_k, Y_i)+b_{ik}\Theta(X_i, Y_k)\}=0.$$

This proves the theorem.

In conclusion, the author wishes to express his deep appreciation to Professor K. Morinaga for his kind guidance.

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ON THE BLOCKS AND THE SECTIONS OF FINITE GROUPS

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(Received September 30, 1960)

1. Let $\mathfrak{G}$ be a group of finite order g and let p be a fixed rational prime; $g = p^a g'$, $(p, g') = 1$. Let us assume that $a > 0$, for otherwise our results will not give any new information. We denote by $\chi_1, \chi_2, \cdots, \chi_n$ the irreducible characters[1] of $\mathfrak{G}$ and by $K_1, K_2, \cdots, K_n$ the classes of conjugate elements in $\mathfrak{G}$; we select a complete system of representatives $G_1, G_2, \cdots, G_n$ for the classes $K_1, K_2, \cdots, K_n$. It is well known that

$$\sum_{i=1}^{n} \chi_i(G_\lambda)\chi_i(G_\mu^{-1}) = 0 \qquad (\lambda \neq \mu) \tag{1.1}$$

and

$$\sum_{\nu=1}^{n} c_\nu \chi_i(G_\nu)\chi_j(G_\nu^{-1}) = 0 \qquad (i \neq j) \tag{1.1'}$$

where c_ν denotes the number of elements in K_ν, $\nu = 1, 2, \cdots, n$.

Let $B_1, B_2, \cdots, B_t$ be the p-blocks[2] of $\mathfrak{G}$ and let $P_1 = 1, P_2, \cdots, P_l$ be a full system of p-elements[3] of $\mathfrak{G}$ such that every p-element of $\mathfrak{G}$ is conjugate in $\mathfrak{G}$ to exactly one element P_δ of the system. It is well known that if we denote by $\mathfrak{S}(P)$ the p-section[4] of a p-element P in $\mathfrak{G}$, then the elements of $\mathfrak{G}$ are distributed into l p-sections $\mathfrak{S}(P_1)$, $\mathfrak{S}(P_2), \cdots, \mathfrak{S}(P_l)$. R. BRAUER, in his paper [1], gave the following refinement of some of the orthogonality relations (1.1):[5]

[1. A] *If L and M are two elements of $\mathfrak{G}$ which belong to different p-sections of $\mathfrak{G}$, then*

$$\sum_{\chi_i \in B_\tau} \chi_i(L)\chi_i(M^{-1}) = 0 \tag{1.2}$$

for each p-block B_τ of $\mathfrak{G}$.

Recently, R. BRAUER and M. OSIMA have given independently a refinement of some of the orthogonality relations (1.1'):

[1. B] *If χ_i and χ_j are two irreducible characters of $\mathfrak{G}$ which belong to different*

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1) The term "irreducible character" will always mean absolutely irreducible ordinary character.
2) Cf. §9 in [3].
3) An element of $\mathfrak{G}$ is called a p-element if its order is a power of p.
4) An element G of $\mathfrak{G}$ can be expressed uniquely as a product PR of two commutative elements where P is a p-element, while R is a p-regular element. We shall call P the *p-factor* of G and R the *p-regular factor* of G. The *p-section* $\mathfrak{S}(P)$ of a p-element P in $\mathfrak{G}$ is the set of all elements of $\mathfrak{G}$ whose p-factors are conjugate to P in $\mathfrak{G}$.
5) This result was announced in [1] and its proof has been printed in [2].

p-blocks of $\mathfrak{G}$, then

(1.3)
$$\sum_{K_\nu \subseteq \mathfrak{S}(P_\delta)} c_\nu \chi_i(G_\nu)\chi_j(G_\nu^{-1})=0$$

for each p-section $\mathfrak{S}(P_\delta)$ of $\mathfrak{G}$. ([9])

Let Ω be the field of g-th roots of unity and let $\mathfrak{p}$ be an arbitrarily fixed prime ideal divisor of p in Ω; $\mathfrak{o}_\mathfrak{p}$ will denote the ring of all $\mathfrak{p}$-integers in Ω. In [11], P. ROQUETTE considered arithmetically the blocks of the character ring of $\mathfrak{G}$ over the ring of all $\mathfrak{p}$-addic integers; K. SHIRATANI has shown in [12] that, essentially, we have only to consider the character ring $X_\mathfrak{p}$ of $\mathfrak{G}$ over $\mathfrak{o}_\mathfrak{p}$. The object of this note is to consider the duals of some results on p-blocks and p-sections, especially the duals of [1. A] and [1. B], from a certain standpoint that (1.1) and (1.1′) are dual. In the duality, the p-regular sections[6] of $\mathfrak{G}$ will correspond to the p-blocks of $\mathfrak{G}$. In § 3, we shall define a new kind of blocks (p-complementary blocks) of group characters which will correspond to p-sections.

2. In this section, we shall sketch an outline of some well known results[7] on $X_\mathfrak{p}$ which will correspond to some fundamental properties[8] of the primitive idempotents of the center $Z_\mathfrak{p}$ of the group ring of $\mathfrak{G}$ over $\mathfrak{o}_\mathfrak{p}$. In the first place, in order to see "dual", we shall refer to the definition of the primitive idempotents of $Z_\mathfrak{p}$.

We denote by Z the center of the group ring of $\mathfrak{G}$ over Ω. The primitive idempotent e_i of Z belonging to χ_i is given by

(2.1)
$$e_i=\frac{1}{g}\sum_{\nu=1}^{n} x_i\chi_i(G_\nu^{-1})K_\nu{}^{9)},$$

where $x_i=\chi_i(1)$. After M. OSIMA [8], we set

(2.2)
$$E_\tau=\sum_{\chi_i\in B_\tau} e_i=\sum_{\nu=1}^{n} b_\nu^\tau K_\nu \qquad (\tau=1,2,\cdots,t).$$

It is well known that all E_τ belong to $Z_\mathfrak{p}$ and that if, for a set B of irreducible characters χ_i of $\mathfrak{G}$, $\sum_{\chi_i\in B} e_i$ belongs to $Z_\mathfrak{p}$, then B is a collection of p-blocks B_τ of $\mathfrak{G}$. Thus we have the following:

[2. A] *In the center $Z_\mathfrak{p}$ of the group ring of $\mathfrak{G}$ over $\mathfrak{o}_\mathfrak{p}$,*

$$1=E_1+E_2+\cdots+E_t$$

is the decomposition of the identity 1 *into the sum of mutually orthogonal primitive idempotents.* ([8])

6) The term "p-regular section" has the same meaning as "$\mathfrak{p}$-Oberklasse" in [11]: The p-regular section $S(R)$ of a p-regular element R in $\mathfrak{G}$ is the set of all elements of $\mathfrak{G}$ whose p-regular factors (See footnote 4) are conjugate to R in $\mathfrak{G}$.

7) Cf. [11], [12].

8) Cf. [8].

9) Each class K_ν is interpreted very often as the sum of all its elements.

We now consider the character ring X of $\mathfrak{G}$ over Ω. Let $d_1, d_2, \cdots, d_n$ be the primitive idempotents of X: $d_\mu(G_\nu)=1$ or 0 according as $\mu=\nu$ or not. If we denote by ξ_μ the linear character of X belonging to d_μ, then $\xi_\mu(\chi_i)=\chi_i(G_\mu)$ for all χ_i. As is easily seen, each d_μ is expressed as

$$d_\mu=\frac{1}{g}\sum_{i=1}^{n} c_\mu \chi_i(G_\mu^{-1})\chi_i. \tag{2.1'}$$

Evidently the elements $d_1, d_2, \cdots, d_n$ form a Ω-basis of X: $X=\Omega d_1+\Omega d_2+\cdots+\Omega d_n$. Further we shall consider four subrings of X:

$$\begin{aligned}
X_{\mathbf{P}}&=\mathbf{P}\chi_1+\mathbf{P}\chi_2+\cdots+\mathbf{P}\chi_n,\\
X_p&=X_{\mathbf{P}}\cap X_{\mathfrak{p}}=\mathbf{I}_p\chi_1+\mathbf{I}_p\chi_2+\cdots+\mathbf{I}_p\chi_n,\\
X_{\mathbf{I}}&=\mathbf{I}\chi_1+\mathbf{I}\chi_2+\cdots+\mathbf{I}\chi_n,\\
\Xi_{\mathbf{I}}&=\mathbf{I}d_1+\mathbf{I}d_2+\cdots+\mathbf{I}d_n,
\end{aligned}$$

where $\mathbf{P}$ is the field of all rational numbers, $\mathbf{I}$ is the ring of all rational integers and $\mathbf{I}_p$ is the ring of all rational p-integers.

Let $R_1=1, R_2, \cdots, R_k$ be a full system of p-regular elements of $\mathfrak{G}$ such that every p-regular element of $\mathfrak{G}$ is conjugate in $\mathfrak{G}$ to exactly one element R_γ of the system. If we denote by $S(R)$ the p-regular section of a p-regular element R in $\mathfrak{G}$, then the elements of $\mathfrak{G}$ are distributed into k p-regular sections $S(R_1), S(R_2), \cdots, S(R_k)$. We set

$$\delta_\gamma=\sum_{K_\nu \subseteq S(R_\gamma)} d_\nu=\sum_{i=1}^{n}\alpha_i^\gamma\chi_i, \tag{2.2'}$$

where

$$\alpha_i^\gamma=\frac{1}{g}\sum_{K_\nu \subseteq S(R_\gamma)} c_\nu\chi_i(G_\nu^{-1}). \tag{2.3}$$

As is easily seen, all α_i^γ are rational numbers: $\delta_1\in X_{\mathbf{P}}$. Moreover δ_1 *belongs to* X_p, as is shown in the following.

Let $\mathfrak{P}$ be a p-Sylow subgroup of $\mathfrak{G}$ and let θ_1 be the 1-character of $\mathfrak{P}$. Denoting by θ_1^* the character of $\mathfrak{G}$ induced by θ_1, we have $\theta_1^*\in X_{\mathbf{I}}\cap\Xi_{\mathbf{I}}$ and

$$\theta_1^*(G)\begin{cases}\equiv\theta_1^*(1)=g'\not\equiv 0 \pmod{p} & (G\in S(1)),\\ =0 & (G\notin S(1)).\end{cases}$$

Therefore

$$(\theta_1^*)^{\varphi(p^\alpha)}\equiv\delta_1 \pmod{p^\alpha\Xi_{\mathbf{I}}} \qquad (\alpha=1,2,\cdots),$$

where φ denotes the Euler's function. Hence, by Lemma 2 in [12], we have

$$(\theta_1^*)^{\varphi(p^\alpha)}-\delta_1\in X_{\mathbf{P}}\cap X_{\mathfrak{p}}=X_p$$

for sufficient large α. Since $\theta_1^*\in X_{\mathbf{I}}\subseteq X_p$, we see that $\delta_1\in X_p$.

In consideration of (2.2′) and (2.3), $\delta_1 \in X_p$ yields the congruences

$$\sum_{P\in S(1)} \chi_i(P) \equiv 0 \pmod{p^a} \qquad (i=1,2,\cdots,n) \tag{2.4}$$

in I.

For an arbitrarily fixed p-regular element $R=R_\gamma$ $(1\leqq\gamma\leqq k)$, we consider the normalizer $\tilde{\mathfrak{G}}=\mathfrak{N}(R)$ of R in $\mathfrak{G}$. We denote by $\tilde{g}$ the order of $\tilde{\mathfrak{G}}$ and by $\tilde{\chi}_1, \tilde{\chi}_2, \cdots, \tilde{\chi}_{\tilde{n}}$ the irreducible characters of $\tilde{\mathfrak{G}}$; $\tilde{\omega}_j$ will denote the linear character of the center $\tilde{Z}$ of the group ring of $\tilde{\mathfrak{G}}$ over Ω belonging to $\tilde{\chi}_j$, $j=1,2,\cdots,\tilde{n}$. The idempotent $\tilde{\delta}_R$ of the character ring $\tilde{X}$ of $\tilde{\mathfrak{G}}$ over Ω associated with the p-regular section of R in $\tilde{\mathfrak{G}}$ is given by

$$\tilde{\delta}_R = \frac{1}{\tilde{g}}\sum_{j=1}^{\tilde{n}} \sum_{P\in\tilde{S}(1)} \tilde{\chi}_j(R^{-1}P^{-1})\tilde{\chi}_j,$$

where $\tilde{S}(1)$ is the p-regular section of 1 in $\tilde{\mathfrak{G}}$. If φ is an element of $\tilde{X}$, we denote by φ^* the element of X induced by φ: $\varphi^*(G)=\frac{1}{\tilde{g}}\sum_{X\in\mathfrak{G}}\varphi_0(X^{-1}GX)$ for $G\in\mathfrak{G}$, where φ_0 is the extension of φ to $\mathfrak{G}$ obtained by putting $\varphi_0(G)=0$ for $G\notin\tilde{\mathfrak{G}}$. By Frobenius' theorem on induced characters, we have

$$\delta_\gamma = \tilde{\delta}_R^* = \left\{\sum_{j=1}^{\tilde{n}} \tilde{\omega}_j(R^{-1})\tilde{\alpha}_j^1\chi_j\right\}^*, \tag{2.5}$$

where

$$\tilde{\alpha}_j^1 = \frac{1}{\tilde{g}}\sum_{P\in\tilde{S}(1)} \tilde{\chi}_j(P^{-1}). \tag{2.6}$$

Since all $\tilde{\alpha}_j^1$ are rational p-integers, δ_γ belongs to $X_\mathfrak{p}$. Conversely if, for a collection S of classes K_ι of $\mathfrak{G}$, $\delta=\sum_{K_\iota\subseteq S} d_\iota$ belongs to $X_\mathfrak{p}$, then S is a collection of p-regular sections $S(R_\gamma)$ of $\mathfrak{G}$. This can be seen as follows:

Suppose that $S\cap S(R_\gamma)$ is not vacuous and that $S\not\supseteq S(R_\gamma)$. Then we can select two classes K_α and K_β in $S(R_\gamma)$ such that $K_\alpha\subseteq S$ but $K_\beta\not\subseteq S$. It is easily seen that $\xi_\alpha(\delta)=1$, while $\xi_\beta(\delta)=0$. On the other hand, in general, if K_λ and K_μ are contained in the same p-regular section of $\mathfrak{G}$, then $\xi_\lambda(\chi_\iota)\equiv\xi_\mu(\chi_\iota) \pmod{\mathfrak{p}}$ for all χ_ι. Hence we have $\xi_\alpha(\delta)\equiv\xi_\beta(\delta) \pmod{\mathfrak{p}}$, which yields a contradiction. Therefore S is a collection of p-regular sections of $\mathfrak{G}$.

We thus obtain the following:

[2. A′] *In the character ring* $X_\mathfrak{p}$ *of* $\mathfrak{G}$ *over* $\mathfrak{o}_\mathfrak{p}$,

$$1=\delta_1+\delta_2+\cdots+\delta_k$$

is the decomposition of the identity 1 *into the sum of mutually orthogonal primitive idempotents.* ([11], [12])

Since every δ_γ belongs to $X_\mathfrak{p}$, there exist k but not more than k irreducible

characters χ_i of $\mathfrak{G}$ which are linearly independent (mod $\mathfrak{p}$) ([5]). We see also that two classes K_α and K_β are contained in the same p-regular section $S(K_\lambda)$ of $\mathfrak{G}$ if and only if $\xi_\alpha(\chi_i)\equiv\xi_\beta(\chi_i)$ (mod $\mathfrak{p}$) for all irreducible characters χ_i of $\mathfrak{G}$.

3. Let $q_0=p, q_1, q_2, \cdots, q_f$ be the rational primes included in the group order g. In the first place, in order to introduce the concept of p-complementary blocks mentioned in §1, we shall give some characterizations of the p-sections $\mathfrak{S}(P_\delta)$ of $\mathfrak{G}$.

[3. A] *Two classes K_α and K_β of $\mathfrak{G}$ are contained in the same p-section $\mathfrak{S}(P_\delta)$ of $\mathfrak{G}$ if and only if there exists a chain of classes*

$$K_\alpha, K_\kappa, \cdots, K_\rho, K_\beta \tag{3.1}$$

of $\mathfrak{G}$ such that any two consecutive classes K_μ and K_ν of the chain are contained in a $q_{r(\mu,\nu)}$-regular section of $\mathfrak{G}$, $1\leqq r(\mu,\nu)\leqq f$.

PROOF. As is well known, an element G of $\mathfrak{G}$ is expressed as a product $Q_0Q_1Q_2\cdots Q_f$, where Q_r is the q_r-factor of $G, r=0, 1, 2, \cdots, f$. It is evident that $Q_0Q_1Q_2\cdots Q_{r-1}$ and $Q_0Q_1Q_2\cdots Q_r$ belong to the q_r-regular section of $Q_0Q_1Q_2\cdots Q_{r-1}$ in $\mathfrak{G}$, $r=1, 2, \cdots, f$. Hence if we denote by $K(Y)$ the class K_λ of $\mathfrak{G}$ represented by an element Y, then the classes $K(G)$ and $K(Q_0)$ are connected by a chain as (3.1). Therefore if K_α and K_β are contained in the same p-section of $\mathfrak{G}$, then they are connected by a chain as (3.1).

Conversely, if K_μ and K_ν are contained in a q_r-regular section of $\mathfrak{G}$ $(1\leqq r\leqq f)$, then the p-factors of them are conjugate in $\mathfrak{G}$. Therefore if K_α and K_β are connected by a chain as (3.1), then they are contained in the same p-section of $\mathfrak{G}$. This completes the proof.

We also have the following characterization of p-sections:

[3. B] *The p-sections $\mathfrak{S}(P_\delta)$ of $\mathfrak{G}$ are characterized as the minimal sets $\mathfrak{S}$ of elements of $\mathfrak{G}$ such that (a) every $\mathfrak{S}$ is not vacuous, (b) every $\mathfrak{S}$ is a collection of q-regular sections of $\mathfrak{G}$ for each rational prime q, different from p.*

Now we define the p-complementary blocks of $\mathfrak{G}$. We shall say that two irreducible characters χ_i and χ_j of $\mathfrak{G}$ belong to the same *p-complementary block* of $\mathfrak{G}$ if and only if there exists a chain of irreducible characters

$$\chi_i, \chi_h, \cdots, \chi_m, \chi_j \tag{3.1'}$$

of $\mathfrak{G}$ such that any two consecutive characters χ_u and χ_v of the chain belong to a $q_{r(u,v)}$-block of $\mathfrak{G}$, $1\leqq r(u,v)\leqq f$. Then we can distribute the irreducible characters $\chi_1, \chi_2, \cdots, \chi_n$ of $\mathfrak{G}$ into a certain number of p-complementary blocks $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_s$ of $\mathfrak{G}$.

[3. B'] *The p-complementary blocks $\mathfrak{B}_\tau$ of $\mathfrak{G}$ are characterized as the minimal sets $\mathfrak{B}$ of irreducible characters χ_i of $\mathfrak{G}$ such that (a) every $\mathfrak{B}$ is not vacuous, (b) every $\mathfrak{B}$ is a collection of q-blocks of $\mathfrak{G}$ for each rational prime q, different from p.*

We shall consider the idempotents of X associated with the p-sections of $\mathfrak{G}$ and the idempotents of Z associated with the p-complementary blocks of $\mathfrak{G}$. We set

(3.2) $$\varepsilon_\delta=\sum_{K_\nu\subseteq\mathfrak{S}(P_\delta)} d_\nu=\sum_{i=1}^{n}\beta_i^\delta\chi_i,$$

where

(3.3) $$\beta_i^\delta=\frac{1}{g}\sum_{K_\nu\subseteq\mathfrak{S}(P_\delta)} c_\nu\chi_i(G_\nu^{-1}).$$

By [2. A] and [3. B], we have the following:

[3. C] *All* $p^a\beta_i^\delta$ *are algebraic integers.*[10] *Conversely, if* $\mathfrak{S}$ *is a collection of classes* K_ν *of* $\mathfrak{G}$ *such that all coefficients* β_i *of*

$$p^{a'}\sum_{K_\nu\subseteq\mathfrak{S}} d_\nu=\sum_{i=1}^{n}\beta_i\chi_i$$

are algebraic integers, then $\mathfrak{S}$ *is a collection of* p*-sections* $\mathfrak{S}(P_\delta)$ *of* $\mathfrak{G}$, *where* a' *is a rational integer.*

Similarly we set

(3.2′) $$\Delta_\sigma=\sum_{\chi_i\in\mathfrak{B}_\sigma} e_i=\sum_{\nu=1}^{n} a_\nu^\sigma K_\nu,$$

where

(3.3′) $$a_\nu^\sigma=\frac{1}{g}\sum_{\chi_i\in\mathfrak{B}_\sigma} x_i\chi_i(G_\nu^{-1}).$$

We then have the following:

[3. C′] *All* $p^a a_\nu^\sigma$ *are algebraic integers. Conversely, if* $\mathfrak{B}$ *is a set of irreducible characters* χ_i *of* $\mathfrak{G}$ *such that all coefficients* a_ν *of*

$$p^{a'}\sum_{\chi_i\in\mathfrak{B}} e_i=\sum_{\nu=1}^{n} a_\nu K_\nu$$

are algebraic integers, then $\mathfrak{B}$ *is a collection of* p*-complementary blocks* $\mathfrak{B}_\sigma$ *of* $\mathfrak{G}$, *where* a' *is a rational integer.*

Corresponding to (2.4), by [3. C] we have the congruences

(3.4) $$\sum_{R\in\mathfrak{S}(1)}\chi_i(R)\equiv 0 \pmod{g'} \qquad (i=1,2,\cdots,n)$$

in I.

Let $\hat{\mathfrak{G}}$ be the normalizer $\mathfrak{N}(P)$ of a p-element $P=P_\delta$ in $\mathfrak{G}$, $1\leqq\delta\leqq l$. If $\hat{g}$, $\hat{\chi}_h$, $\hat{\omega}_h$, $\hat{n}$, $\hat{\varepsilon}_P$, $\hat{\varepsilon}_P^*$, $\hat{\mathfrak{S}}(1)$ for $\hat{\mathfrak{G}}$ correspond to $\tilde{g}$, $\tilde{\chi}_j$, $\tilde{\omega}_j$, $\tilde{n}$, $\tilde{\delta}_R$, $\tilde{\delta}_R^*$, $\tilde{S}(1)$ for $\mathfrak{G}=\mathfrak{N}(R)$ in §2, then corresponding to (2.5) we have

(3.5) $$\varepsilon_\delta=\hat{\varepsilon}_P^*=\left\{\sum_{h=1}^{n}\hat{\omega}_i(P^{-1})\hat{\beta}_h^1\hat{\chi}_h\right\}^*,$$

10) Cf. [8].

where

$$\hat{\beta}_h^1 = \frac{1}{\hat{g}} \sum_{R \in \hat{\mathfrak{S}}(1)} \hat{\chi}_h(R^{-1}). \tag{3.6}$$

REMARK. Every Osima's block[11] of $\mathfrak{G}$ for p is a collection of p-complementary blocks of $\mathfrak{G}$. If $\mathfrak{G}$ has a normal p-Sylow subgroup, then the Osima's blocks of $\mathfrak{G}$ for p and the p-complementary blocks of $\mathfrak{G}$ are identical as a whole. But, in general, both concepts of blocks are not identical.

4. In the first place, we shall refer a result on the primitive idempotents E_τ of $Z_\mathfrak{p}$, in order to see "dual" and to use it.

Let P be an arbitrarily given p-element of $\mathfrak{G}$ and let $\hat{\mathfrak{G}}$ denote the normalizer $\mathfrak{N}(P)$ of P in $\mathfrak{G}$. We denote by $\hat{B}^{(\tau)}$ the collection of p-blocks $\hat{B}_\rho$ of $\hat{\mathfrak{G}}$ such that each $\hat{B}_\rho$ determines a given p-block B_τ of $\mathfrak{G}$ in BRAUER's sence[12]; $\hat{E}^{(\tau)}$ will denote the idempotent of the center $\hat{Z}_\mathfrak{p}$ of the group ring of $\hat{\mathfrak{G}}$ over $\mathfrak{o}_\mathfrak{p}$ associated with $\hat{B}^{(\tau)}$, i.e. the sum of all primitive idempotents $\hat{E}_\rho$ of $\hat{Z}_\mathfrak{p}$ such that each $\hat{E}_\rho$ is associated with a p-block $\hat{B}_\rho$ of $\hat{\mathfrak{G}}$ contained in $\hat{B}^{(\tau)}$. Let $\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_{\hat{k}}$ be a complete system of representatives for the p-regular classes of $\hat{\mathfrak{G}}$. We denote by $K(G)$ the class K_μ of $\mathfrak{G}$ represented by an element G and by $\hat{K}(\hat{R}_\gamma)$ the class of conjugate elements in $\hat{\mathfrak{G}}$ represented by $\hat{R}_\gamma$. As is well known, the p-section $\mathfrak{S}(P)$ of P in $\mathfrak{G}$ is the collection of classes $K(P\hat{R}_1)$, $K(P\hat{R}_2), \cdots, K(P\hat{R}_{\hat{k}})$. We then have the following:

[4. A] *For* $\alpha = 1, 2, \cdots, \hat{k}$, *we have*

$$K(P\hat{R}_\alpha)E_\tau = \sum_{\beta=1}^{\hat{k}} b_{\alpha\beta}^\tau K(P\hat{R}_\beta)$$

and

$$\hat{K}(\hat{R}_\alpha)\hat{E}^{(\tau)} = \sum_{\beta=1}^{\hat{k}} b_{\alpha\beta}^\tau \hat{K}(\hat{R}_\beta)$$

with the same coefficients $b_{\alpha\beta}^\tau$. ([5])

Let q be an arbitrary rational prime different from p and let Q be an arbitrarily given q-element of $\mathfrak{G}$. We select a complete system of representatives $T_1, T_2, \cdots, T_m$ for the q-regular classes of the normalizer $\mathfrak{N}(Q)$ of Q in $\mathfrak{G}$. We denote by $\mathfrak{B}^{(\sigma)}(Q)$ the collection of q-blocks of $\mathfrak{N}(Q)$ such that each q-block of the collection determines a q-block of $\mathfrak{G}$ contained in a given p-complementary block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$; $\Delta^{(\sigma)}(Q)$ will denote the idempotent of the center $Z(Q)$ of the group ring of $\mathfrak{N}(Q)$ over Ω associated with $\mathfrak{B}^{(\sigma)}(Q)$. By [4. A] we may write

$$K(Q)\Delta_\sigma = \sum_{\beta=1}^{m} a_\beta^\sigma(Q) K(QT_\beta)$$

11) Cf. [7], [6].

12) Cf. [1], [2]. If a p-block B_τ of $\mathfrak{G}$ is determined by a p-block $\hat{B}_\rho$ of $\hat{\mathfrak{G}}$, then B_τ is denoted by $\hat{B}_\rho^{\mathfrak{G}}$ in the notations of [2].

and

$$\Delta^{(\sigma)}(Q)=\sum_{\beta=1}^{m}a_{\beta}^{\sigma}(Q)K(Q;\,T_{\beta})$$

with the same coefficients $a_{\beta}^{\sigma}(Q)$, where $K(Q;T_{\beta})$ denotes the class of conjugate elements in $\mathfrak{N}(Q)$ represented by T_{β}. Since all $p^{a}a_{\beta}^{\sigma}(Q)$ are algebraic integers, from [3. C′] we see that every $\mathfrak{B}^{(\sigma)}(Q)$ is a collection of p-complementary blocks of $\mathfrak{N}(Q)$.

Let R be an arbitrarily given p-regular element of $\mathfrak{G}$ and let Q_r be the q_r-regular factor of $R, r=1,2,\cdots,f$. Let further $\tilde{P}_1,\tilde{P}_2,\cdots,\tilde{P}_{\tilde{l}}$ be a complete system of representatives for the classes of conjugate elements in the normalizer $\tilde{\mathfrak{G}}=\mathfrak{N}(R)$ of R in $\mathfrak{G}$ which are contained in the p-regular section $\tilde{S}(1)$ of 1 in $\tilde{\mathfrak{G}}$; $\tilde{K}(\tilde{P}_{\delta})$ will denote the class of conjugate elements in $\tilde{\mathfrak{G}}$ represented by $\tilde{P}_{\delta}$. For each p-complementary block $\mathfrak{B}_{\sigma}$ of $\mathfrak{G}$, we can define the collection $\tilde{\mathfrak{B}}^{(\sigma)}$ of p-complementary blocks $\tilde{\mathfrak{B}}_{\rho}$ of $\tilde{\mathfrak{G}}$ such that each $\tilde{\mathfrak{B}}_{\rho}$ *determines* $\mathfrak{B}_{\sigma}$, in the same way as in § 3 of [7]. In consequence of this, we have the following theorems.

THEOREM 1. *For $\alpha=1,2,\cdots,\tilde{l}$, we have*

$$K(R\tilde{P}_{\alpha})\Delta_{\sigma}=\sum_{\beta=1}^{\tilde{l}}a_{\alpha\beta}^{\sigma}K(R\tilde{P}_{\beta})$$

and

$$\tilde{K}(\tilde{P}_{\alpha})\tilde{\Delta}^{(\sigma)}=\sum_{\beta=1}^{\tilde{l}}a_{\alpha\beta}^{\sigma}\tilde{K}(\tilde{P}_{\beta})$$

with the same coefficients $a_{\alpha\beta}^{\sigma}$, where $\tilde{\Delta}^{(\sigma)}$ is the idempotent of the center $\tilde{Z}$ of the group ring of $\tilde{\mathfrak{G}}$ over Ω associated with $\tilde{\mathfrak{B}}^{(\sigma)}$.

THEOREM 2. *If L and M are two elements of $\mathfrak{G}$ which belong to different p-regular sections of $\mathfrak{G}$, then*

$$\sum_{\chi_i\in\mathfrak{B}_{\sigma}}\chi_i(L)\chi_i(M^{-1})=0$$

for each p-complementary block $\mathfrak{B}_{\sigma}$ of $\mathfrak{G}$.[13]

[4. B] *If*

$$\sum_{\nu=1}^{n}a_{\nu}K_{\nu}\Delta_{\nu}=0 \qquad (a_{\nu}\in\Omega),$$

then

$$\sum_{K_{\nu}\subset S(R_{\gamma})}a_{\nu}K_{\nu}\Delta_{\nu}=0$$

for each p-regular section $S(R_{\gamma})$ of $\mathfrak{G}$.

13) This is the refinement mentioned in footnote 6) of [6].

THEOREM 3. *If χ_i and χ_j are two irreducible characters of $\mathfrak{G}$ which belong to different p-complementary blocks of $\mathfrak{G}$, then*

$$\sum_{K_\nu \subseteq S(R_\gamma)} c_\nu \chi_i(G_\nu)\chi_j(G_\nu^{-1})=0$$

for each p-regular section $S(R_\gamma)$ of $\mathfrak{G}$.

It was shown in [8] that each E_τ is a linear combination of the p-regular classes K_ν of $\mathfrak{G}$, i. e. the classes K_ν of $\mathfrak{G}$ contained in $\mathfrak{S}(1)$. Moreover, [4. A] shows that if K_α is contained in a p-section $\mathfrak{S}(P_\delta)$, then $K_\alpha E_\tau$ is a linear combination of the classes K_β contained in $\mathfrak{S}(P_\delta)$. (1.3) shows that if χ_i belongs to a p-block B_τ of $\mathfrak{G}$, then $\chi_i \varepsilon_\delta$ is a linear combination of the characters χ_j belonging to B_τ; especially ε_δ is a linear combination of the characters χ_j which belong to the p-block B_1 of $\mathfrak{G}$ containing the 1-character χ_1. Theorems 1 and 3 imply the corresponding results for the p-complementary blocks and the p-regular sections of $\mathfrak{G}$.

REMARK. Let Π be a set of rational primes included in g and let $\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_u$ be the minimal sets of irreducible characters χ_i of $\mathfrak{G}$ such that (a) every $\mathbf{B}_\lambda$ is not vacuous, (b) every $\mathbf{B}_\lambda$ is a collection of q-blocks of $\mathfrak{G}$ for each $q \in \Pi$ and let $\mathbf{S}_1, \mathbf{S}_2, \cdots,$ $\mathbf{S}_v$ be the minimal collections of classes K_ν of $\mathfrak{G}$ such that (a) every $\mathbf{S}_\mu$ is not vacuous, (b) every $\mathbf{S}_\mu$ is a collection of q-regular sections of $\mathfrak{G}$ for each rational prime q outside of Π (which is included in g); we shall call the sets $\mathbf{B}_\lambda$ of χ_i the *Π-blocks* of $\mathfrak{G}$ and the collections $\mathbf{S}_\mu$ of K_ν the *Π-sections* of $\mathfrak{G}$. By making use of the Π-blocks and Π-sections of $\mathfrak{G}$, we can generalize Theorems 2 and 3.

5. As an application of (2.2′) and (2.3), we shall determine the primitive idempotents of the character ring $\mathbf{X}_p$ of $\mathfrak{G}$ over $\mathbf{I}_p$.[14]

For an arbitrarily given p-regular element R of $\mathfrak{G}$, we shall use again the notations $\tilde{\mathfrak{G}}, \tilde{n}, \tilde{\chi}_j, \tilde{\omega}_j, \tilde{\alpha}_j^1$ used in §2. Let $C_{(R)}$ be the p-elementary class attached to R, i.e. the collection of p-regular sections $S(R_\lambda)$ of $\mathfrak{G}$ such that each R_λ is conjugate in $\mathfrak{G}$ to R^y for a rational integer y prime to the order r of R. We may assume that each R_λ with $S(R_\lambda) \subseteqq C_{(R)}$ is a power of R; $\tilde{\mathfrak{G}}$ is the normalizer of each R_λ with $S(R_\lambda) \subseteqq C_{(R)}$ in $\mathfrak{G}$. We see from (2.3) that if $S(R_\lambda) \subseteqq C_{(R)}$, then

$$\alpha_i^\lambda = \sum_{j=1}^{\tilde{n}} r_{ij}\tilde{\alpha}_j^1 \tilde{\omega}_j(R_\lambda^{-1}) \tag{5.1}$$

where the r_{ij} are defined by

$$\chi_i(\tilde{G}) = \sum_{j=1}^{\tilde{n}} r_{ij}\tilde{\chi}_j(\tilde{G}) \qquad (\tilde{G} \in \tilde{\mathfrak{G}}). \tag{5.2}$$

Hence if we set

$$\delta_{(R)} = \sum_{S(R_\lambda) \subseteqq C_{(R)}} \delta_\lambda = \sum_{i=1}^{n} \alpha_i^{(R)} \chi_i, \tag{5.3}$$

then

14) K. SHIRATANI has determined the primitive idempotents of X_p in his paper [12].

$$\alpha_i^{(R)} = \sum_{j=1}^{\tilde{n}} r_{ij}\tilde{\alpha}_j^1 \left\{ \sum_{S(R_\lambda)\subseteq C_{(R)}} \tilde{\omega}_j(R_\lambda^{-1}) \right\}. \tag{5.4}$$

We denote by G the galois group of the field of g'-th roots of unity over $\mathbf{P}$. If $\sigma\in G$, then for each R_λ with $S(R_\lambda)\subseteq C_{(R)}$ there exists a rational integer y_σ prime to r such that $\sigma(\tilde{\omega}_j(R_\lambda))=\tilde{\omega}_j(R^{y_\sigma})$ for $j=1,2,\cdots,\tilde{n}$. By the above assumption, R^{y_σ} is written as $R_{\lambda(\sigma)}$; $\lambda(\sigma)$ will be determined uniquely. From (5.1) we see that $\sigma(\alpha_i^\lambda)=\alpha_i^{\lambda(\sigma)}$ for all χ_i.

We consider similarly for each p-elementary class of $\mathfrak{G}$. Then the substitutions $\lambda\to\lambda(\sigma)$ define a permutations group on k linearly independent n-dimensional vectors

$$\mathfrak{a}_1=(\alpha_1^1, \alpha_2^1, \cdots, \alpha_n^1),\ \mathfrak{a}_2=(\alpha_1^2, \alpha_2^2, \cdots, \alpha_n^2),\ \cdots,\ \mathfrak{a}_k=(\alpha_1^k, \alpha_2^k, \cdots, \alpha_n^k)$$

by putting $\sigma(\mathfrak{a}_\lambda)=\mathfrak{a}_{\lambda(\sigma)}$, σ running over all galois substitutions of G. For each p-elementary class $C_{(R)}$ of $\mathfrak{G}$, the vectors $\mathfrak{a}_\lambda$ associated with the p-regular sections $S(R_\lambda)$ with $S(R_\lambda)\subseteq C_{(R)}$ form a set of transitivity. Combining this fact with [2. A′], we see that for a collection C of classes K_ν of $\mathfrak{G}$, $\sum_{K_\nu\subseteq C} d_\nu$ belongs to X_p if and only if C is a collection of p-elementary classes of $\mathfrak{G}$. Therefore if we assume that $C_{(R_1)}, C_{(R_2)}, \cdots, C_{(R_e)}$ are the p-elementary classes of $\mathfrak{G}$, then we have the following:

[5. A] *In the character ring X_p of $\mathfrak{G}$ over the ring $\mathbf{I}_p$ of rational p-integers,*

$$1=\hat{o}_{(R_1)}+\hat{o}_{(R_2)}+\cdots+\hat{o}_{(R_e)}$$

is the decomposition of the identity 1 *into the sum of mutually orthogonal primitive idempotents.* ([12])

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STRUCTURE, COMPOSITION AND FORMATION-MECHANISM OF "SARA-ISHI"

Tadashi OKAHATA

(Received August 1, 1960)

Abstruct

By observing the actual state of Sara-ishi in its place of production, Mt. Aso, the author confirms that the characteristic marginal projection of Sara-ishi hangs orginally towards the ground as was stated by Namba,[1] and also makes clear the reasons why two sides of Sara-ishi give different appearances with each other.

Further, he explains the internal structure, especially the structure and composition of the crust part of the Sara-ishi by investigating it with microscopic observation, with inspection by X-ray, with chemical analysis and with measurement of magnetism. As previously stated,[2] Sara-ishi consists of two parts, i. e., the core part and the crust part. The crust part is formed of volcanic ashes of Mt. Aso, so called Yona, which firstly cover the surface of the core-part substance, and then coagulate there or at the margin of it in the form of the hanging rim acted by the colloidal silicic anhydride contained in Yona. And he confirms that these coagulations take place, by a sort of weathering, after Yona was accumulated on the surface of the core-part substance.

I. Introduction

Sara-ishi is found only at the Mt. Aso. It was named "Lava-cake" by T. Hiki,[3] but from the likness in shape to dish, it is commonly called Dish-stone or Sara-ishi in Japanese. M. Namba reported in his paper[1] that it is found mainly at the ancient encircling-crater mountain of Nakadake, especially Sara-yama, at the surroundings of Kishimadake and at Ōjōdake, but as is shown in Fig. I, the author was able to discover it on the eastern, southern and northern sides of Takadake, at Saishi-ga-hana, a part of inner wall of the old somma north-east of Takadake, and at another part of the inner wall of the old somma, south-east of Takadake, too.

So far, several authors[1,4,5,6,7] have studied about Sara-ishi, but all of them except M. Numba have considered that Sara-ishi is a kind of volcanic bombs which was produced primarily by the eruption of the volcano. They said that when some magma which were hurled up into the atmosphere by the eruption, fell down and stroke agaist the ground, its marginal part, by the impulsive action, was lifted up, so that suddenly a dish-like block of lava was formed.

On the contrary, M. Namba considered that Sara-ishi was formed by the fine pieces of Yona sticked and coagulated by some binder on some of the pre-existing rock fragments which had been laid on the ground. Especially, he believed that the characteristic projection at the margin of Sara-ishi was not formed mechanically by raising itself upward against the ground, but was formed by grewing its edge gradually toward

Fig. 1 The habitats of Sara-ishi.
1. Kishima-dake 2. Ōjō-dake 3. Narao-dake 4. Taka-dake
5. Naka-dake 6. Sara-yama 7. Saishi-ga-hana

the ground.

Hereupon, it becames important problem for the investigation of the origin of Sara-ishi to determine the original direction of the rim, that is, whether it is raised upwardly against the ground or it grows downwardly toward the ground. In other words, it is important to discriminates the upper and lower side of Sara-ishi.

The author minutely observed the actual state of the Sara-ishi on Mt. Aso, collected a large number of them and made clear the reasons why the two sides of Sara-ishi give different appearances with each other. At the same time, investigating the internal structure of Sara-ishi with microscopic observation, with inspection by X-ray, with chemical analisis and with measurement of magnetism, the author studied essential nature of Sara-ishi and manifested its mechanism of production. The outlines of the results are as follows.

II. Observation of the External Appearance and the Configuration of Sara-ishi

One side of Sara-ishi is black or dark brown and has a gloss of some kind, and

gives comparatively compact and smooth feeling, as if the surface of the natural lava was painted with certain paints in some degree, on the other hand, another side is yellowish brown or light gray (ash color) and has no gloss and gives coarse and rough feeling. On this side, there exist some roundish small prominences and stalactitic processes, which exist frequently in groups at the part of large hollow. The edge of this side is lifted like the rim of tray. This gives rise to the name of Sara-ishi.

As to the forms of Sara-ishi, Harada[6] grouped into four standard types, but according to the result of the author's observation with a large number of Sara-ishi, it is founded that there are many samples which can not be classified into these types. Sara-ishi has so many variety of forms, that it should be very difficult to classify them into several types, even though it could be. Rather it seems to be absurd.

In addition, as for the size of Sara-ishi, Harada stated that it should be 2～10cm in diameter, but from the formative and constructive point of view, there are many stones and rocks which should be called in the name of Sara-ishi. They have large amount of varieties in size. Large one amounts to meters and small one doesn't reach even two centimeters. The author is even of opinion that, it is not exaggerated to say that the rocks and stones which are covering the ancient encircling-crater mountain of Nakadake, particularly Sara-yama are entirely Sara-ishi. Here author shows the photographs of the typical examples of Sara-ishi in Figs. 2～7 and Figs. 2′～7′. Figs. 2～7 show the dark or dark brown side of Sara-ishi, viz. glossy side and Figs. 2′～7′ show the side hemmed by the projection each of which corresponds to each of Figs. 2～7 respectively. These photographs show obviously the difference between two sides.

Still more, the author confirmed by minutely observing the actual place of production of Sara-ishi, that the rim of it hangs toward the ground. That is to say. comparatively large-sized Saraishi which is seemed to have been suffered no special change either in naturally or in artifically concerning the up-and-down direction of it, has a rim, without exception, hanging toward the ground, and no rim projecting upward. And he also confirmed that the stalactitic prominences alway hang in the same direction with the hanging rim, too.

Particularly, at the end of the old lava-flow which have not been moved vertically concerning the up-and-down direction of it, author found the hanging rim similar to the one in Sara-ishi, and at the same time, he found the stalactitic processes hanging in group from the ceiling of the cave which existed in this place of lava-flow. (Fig. 8)

Hereupon, it can be said that the projecting rim and stalactitic processes of Sara-ishi always hang toward the ground in natural. Now, it is reasonable to call the side contacting with the ground when Sara-ishi is forming, in the name of "the lower side", and the opposite side, "the upper side". In other words, natural Sara-ishi should be found in the state placing the inside of the "Sara" down and the outside of the "Sara" up. The inside of the "Sara" is the lower side of Sara-ishi and the outside is the upper side of it. In the following, the author will use these definitions. Of course, these definitions are diametrically opposite to Harada's.

Fig. 9 shows the volcanic bomb usually called as the name of the "Maru-shō-hachi", which means the round volcanic bomb hurled up at the eruption in 1933 (8-th year of Showa era). It is confirmed that it has not been moved since the time of its falling,

Fig. 2

Fig. 3'

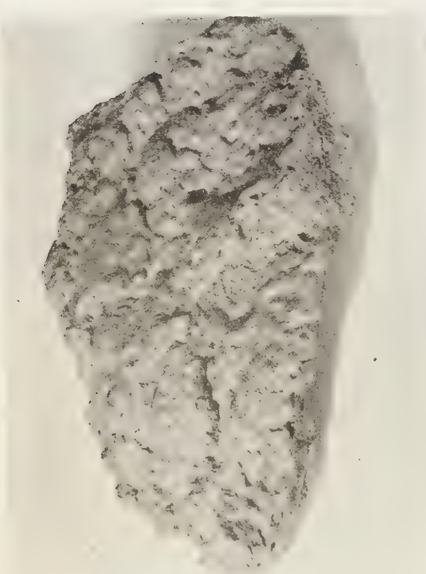


Fig 2

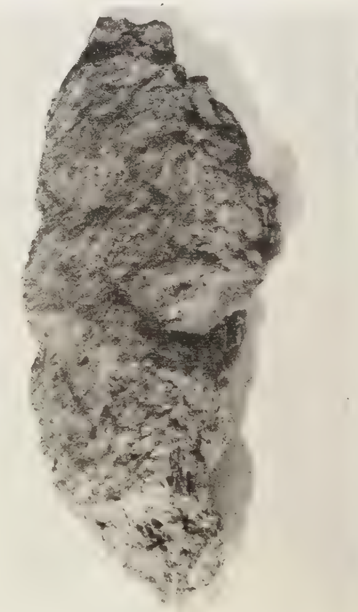


Fig. 3

Fig. 4

Fig. 5

Fig. 1

Fig. 5

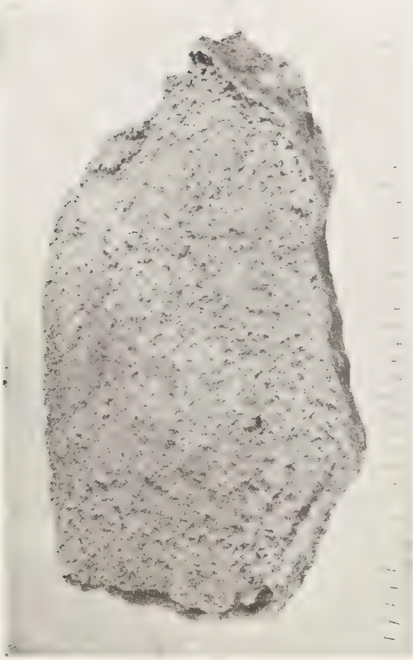


Fig. 6'

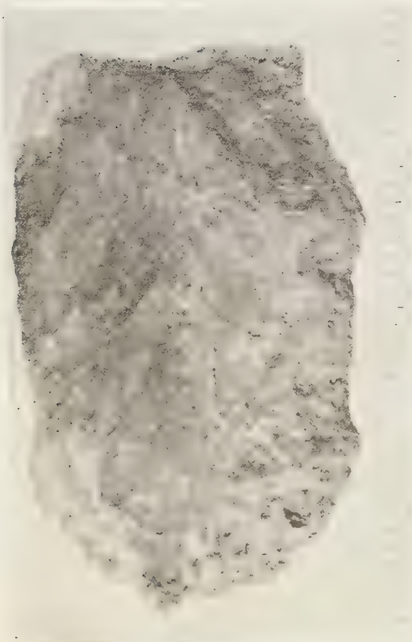


Fig. 6

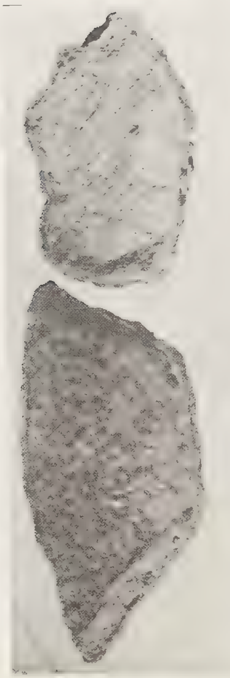
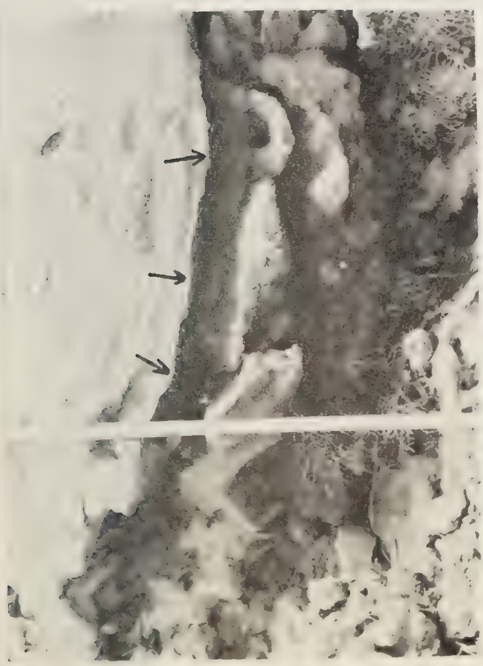


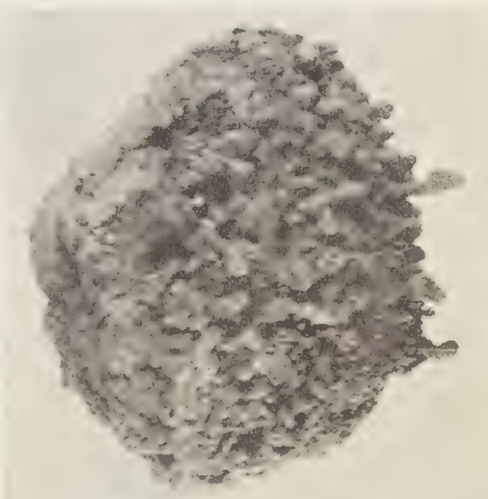
Fig. 7'

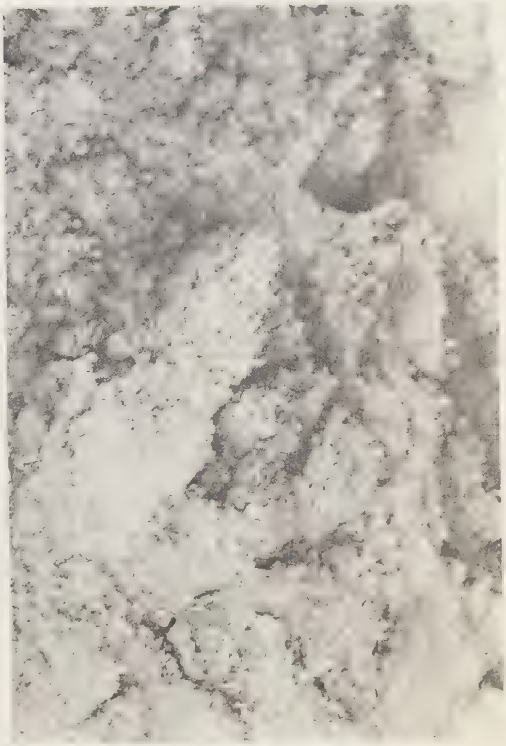
Fig. 7



(i) Hanging rim at the end of lava-flow.

(ii) The cave at the end of lava-flow





(iii), (iv) Stalactitic processes hanging from the ceiling of the cave

Fig. 8

Fig. 9 The volcanic bomb named "Maru-sho-hachi".

0^{hr} 30^{min} a. m., March 1st, 1933. After that, it has not been moved. Its surface is reddish dark brown and has a gloss. It seems as if it is lightly painted on the texture of a natural lava, and explains the state of unevenness of the texture. On the other hand, the side facing to the ground shows yellowish brown color and has no gloss; its rim looks like the projecting rim of Sara-ishi, and on this side hangs many roundish small prominences. Accordingly this bomb "Maru-shō-hachi" can not only be considered as a sort of Sara-ishi, but also as a proof to the fact that the marginal part of Sara-ishi is formed by growing its rim toward the ground.

III. Structure and Composition of Sara-ishi

§ 1. Structure and Composition of the Crust Part of Sara-ishi

i) Microscopic observation as previously stated[(2)], Sara-ishi is made up of the crust part and the core part.

The crust part of a specimen which was cut out of Sara-ishi by the XZ-plane was examined under the microscope. The crust part of Sara-ishi is found to enclose the core part with the thickness of 0.5~3 mm. In the crust part the reddish brown stripes or flow lines pile one over another and run around the core. Fig. 11 (a) and (b) are the

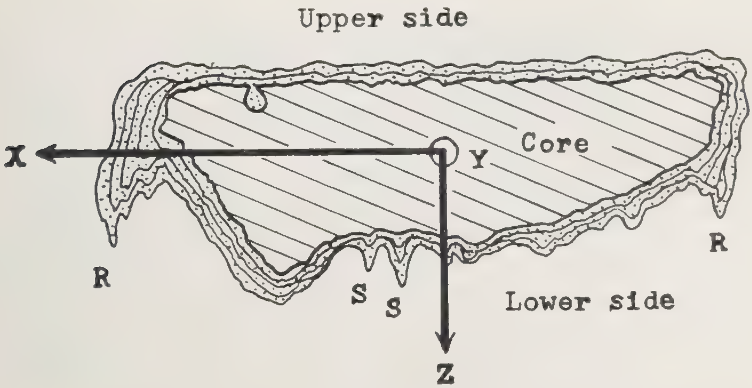


Fig. 10 Schematic repersentation of Sara-ishi

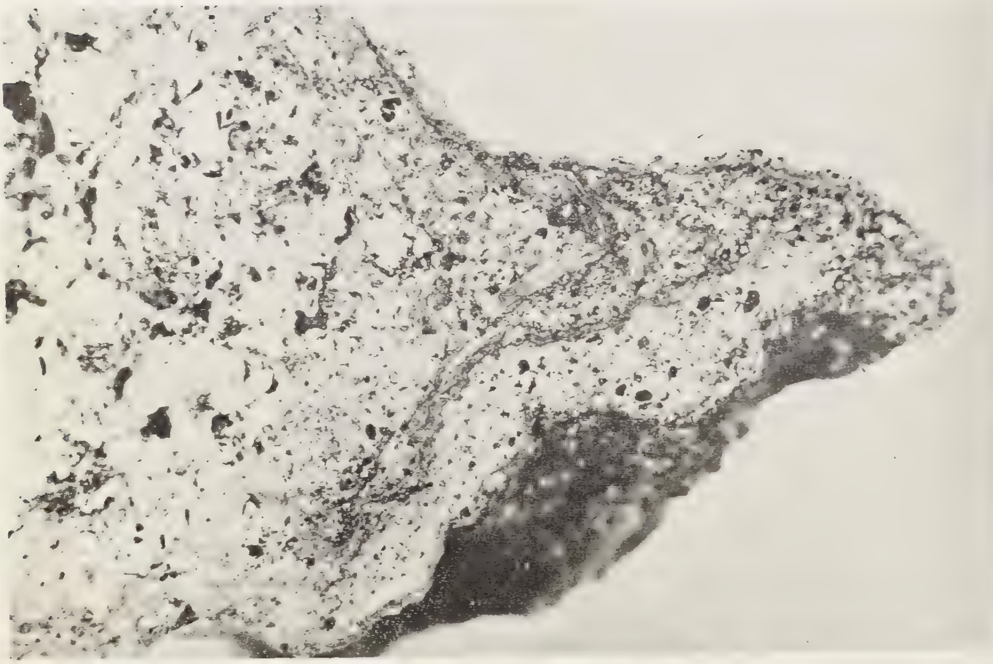
microscopic photographs of the XZ–plane of hanging part of Sara-ishi, and show evidently the layer of the flow lines. The existence of these flow lines is characteristic to the crust part of Sara-ishi, and its reddish and brown appearance is due to ferric oxide. Particularly, in the stalactitic processes, these flow lines run concentrically in the XY–plane like an annual ring.

The ground mass of the crust part is made up of the amorphous substances. There are small crystals of plagioclase, common augite and magnetite in it[illegible], but it does not contain olivine, hypersthene and silicic anhydride crystal, so called quartz. Further, the kinds of the mineral crystals contained in the crust part are quite the same as those contained in Yona[8].

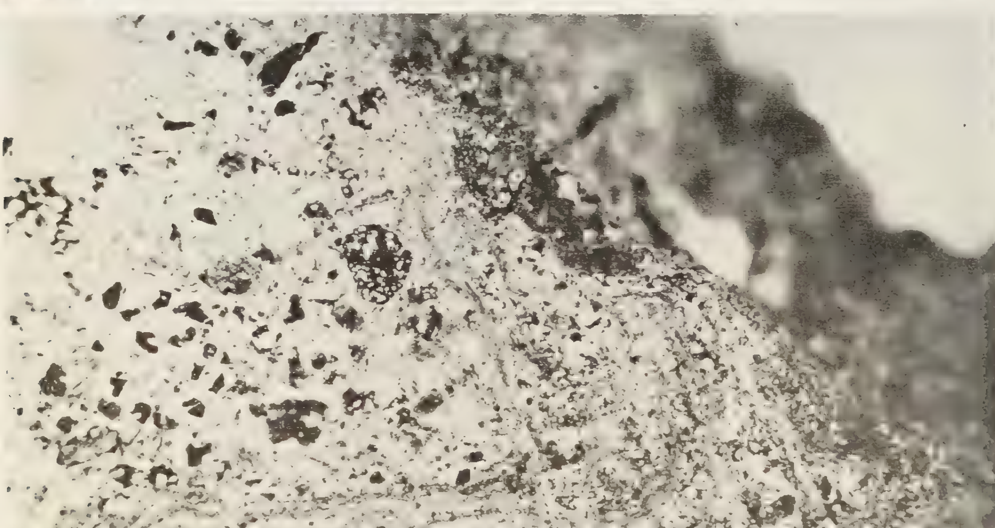
ii) Inspection by X–ray

In order to clarify the substances constructing the crust part, the author firstly took the X–ray powder photographs of those constructing its hanging part, and obtained the ones as shown in Figs. 12 (a) and (b). Fig. 12 (a) was photographed by the X–ray beam from the Cu-anticathode and Fig. 12 (b) was taken by the X–ray beam from the Mo-anticathode. From these photographs, the author obtained the same results as was obtained by the microscopic observation previously mentioned. That is, it is reaffirmed that there are crystals of plagioclase, common augite and magnetite, but quartz does not exist. This constitution of the crust part is in accord with the results previously obtained by examining the Laue-photograph of it[illegible]. Thus, the author obtained the further evidence that the components of the crust part are coincide with those of Yona.

Next, in order to get the informations about the size and the crystal arrangement of the constituents of the crust part, the author examined Laue-photographs. To the thin-leaf specimen which was cut out of the hanging rim of Sara-ishi by the XZ plane, the X–ray beam was projected from Y-direction and the photographs were taken. As is easily recognized from Fig. 13, these photographs show a peppering of diffraction spots and blurred D-S rings. This is due to the disordered arrangements of the various mineral crystalline particles of various size in the amorphous ground mass of the crust part. Then, on the thin-leaf specimens which were cut out of the stalactitic process by



(b)



(a)

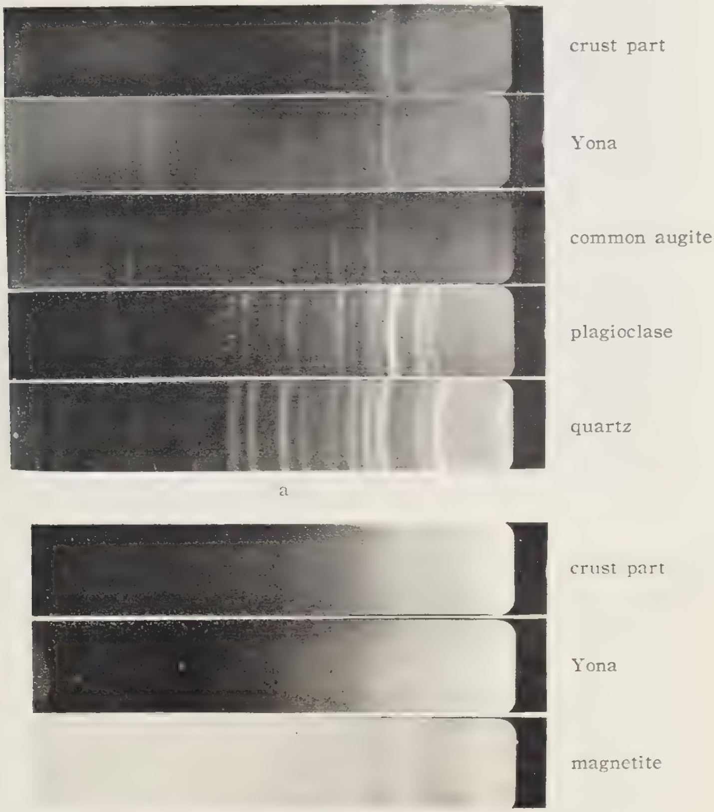


Fig. 12 Powder photographs.

the XZ-plane and the XY-plane, the X-ray beam was projected from Y-and Z-direction respectively, and Laue-photographs were taken. In both cases, the Laue-pattern as in Fig. 14 appeared. Comparing Fig. 13 and Fig. 14, it can be marked that there are much more fine particles in this part than in the crust part.

Further, the presence of the fibrous structure of the stalactitic process was examined by taking the X-ray diffraction photographs in the following way. Namely, to the specimen which was cut out of the stalactitic process by the XZ-plane, the X-ray beam was projected from Y-direction; then the specimen was rotated little by little by small angles around the X-axis and at each time the photographs of the diffraction patterns were taken. Next, it was rotated little by little by small angles around the Z-axis too, and at each time was photographed in the same manner. The same operations were repeated with the XY-cut specimen. But in all of above cases, the almost same kind of patterns as Fig. 13 or Fig. 14 were obtained and none of the photographs showed the fibrous structure. Therefore, it was ascertained that the stalactitic process, as well as the other portion belonging to the crust part, are formed with the disorderly coagulated

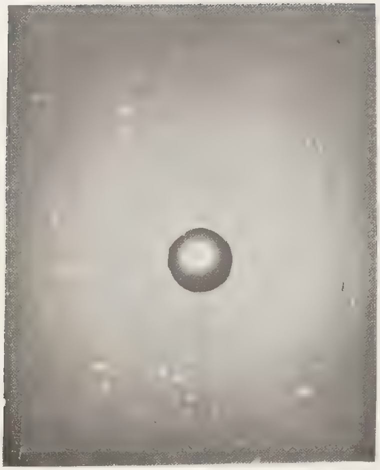


Fig. 13 Laue photograph obtained from the rim of Sara-ishi.

Fig. 14 Laue photograph obtained from the stalactitic process of Sara-ishi.

amorphous substances and fine particles of crystalline minerals, and have not fibrous structure.

§ 2. Structure and Composition of Core Part and Boundary between Core and Crust Parts

i) Structure and Composition of Core Part

As previously stated, main core part substances are volcanic bombs, lapilli and blocks of tuff-like lava, all of which are ejecta. Many of them are porous and their apparent specific gravities are mostly 1.2 ~1.5 and some of them are 1.8 or more. It seems that these pores are the marks of the exhaustion of gases from the volcanic bomb in its course of cooling. The layer of flow-lines which is found in the crust can never be found in their core part.

As is already reported, the ground mass of the core part is vitrious and phenocrysts of plagioclase, common augite, hypersthene and magnetite etc. are distributed irregularly in it.

By heating a fragment of Sara-ishi intensely using a hand burner, the author confirmed the fact that when the crust part turns red, cracks and becomes more or less brittle, the core part is already fused and there yields a sort of many small glass balls. This fact means that the core part is more fusible than the crust part as was reported previously by M. Namba.[1]

ii) Boundary between Core and Crust Part

Observing the boundary between the core part and the crust part minutely, it is found that some of the pores lying close to the surface of the core part are filled up by the same substance as the crust part.

§ 3. Coagulated Substance of Crust Part

As above stated, the crust part are coagulated around the core part. This coagulation

is of course due to the action of a certain material contained in the crust part substance.

About the chemicall components of the crust part substance were already reported by the author[8]. Among those components, the ones which are supposed to be responsible for the coagulation are silicic anhydride, gypsum and ferric oxide.

Now, indeed the silicic anhydride is the main component of the crust part substances and its quantity comes up to about 30 % (wt) but, as above stated, the crust part does not contain the silicic anhydride crystal, so called quartz. Accordingly, it can be said that the free silicic anhydride in the crust part is not of crystalline structaure but is of what is called the colloidal state. In other words, the main material constructing the ground mass of the crust part is the colloidal silicic anhydride. From the chemical point of view, it is natural to consider that the colloidal silicic anhydride does contribute to cause the coagulation.

Next, as was reported previously, the amount of gypsum is so small, that the contribution to the coagulation must be very small.

Finally, if the coagulation of the crust part were due to the ferric oxide, then, the crust part should be broken down by dissolving away the ferric oxide. However, it is found that, when the piece of the crust part is treated by the aqua regia to dissolve away the ferric oxide, it preserves its original form and does not break down.

The fact that the coagulation of the crust part is certainly due to the colloidal silicic anhydride is also confirmed by the following fact that, as soon as crust part of Sara-ishi is treated by caustic soda, it becomes muddily and break down.

The colloidal silicic anhydride, the most important component for the coagulation of the crust part, is supposed to be produced at the bottom of the Aso-crater and scattered over the ground as an ingredient of Yona[1].

Moreover, the colloidal silicic oxide can be supposed to be produced also in the Yona accumulated on the ground in the following way.

As the Yona has not only the strong adhesive property but also the remarkable hygroscopic one and has a tendency of causing sintering phenomena easily, when accumulated even on the surface of a rock on the ground, it keeps to stay there. Therefore, during the time it is exposed in the volcanic atmosphere containing SO_2, it can cause a sort of hydrothermal reaction and can yields colloidal silicic oxide newly in its inner part.

IV. Magnetism of Sara-ishi

The magnetism of Sara-ishi is investigated by using astatic needle. The apparatus is shown schematically in Fig. 15. In the figure, T, M and S_p denote the telescope with scale, the astatic needle attached by the small mirror and Sara-ishi fixed on the holder, respectively. The values of R and r are 200cm and 12cm.

The specimen used is so chosen as to be considered as approximately a disk, that is, it is comparatively flat, the diameter of which is more than twice as its thickness, and when projected on the plane, the maximum diameter of which is less than the twice of its minimum diameter. The specimen holder can be rotated around the vertical axis, the rotating angle of which can also be measured. The specimen is fixed almost horizontally on the holder placing its properly upper surface up and coinciding its center

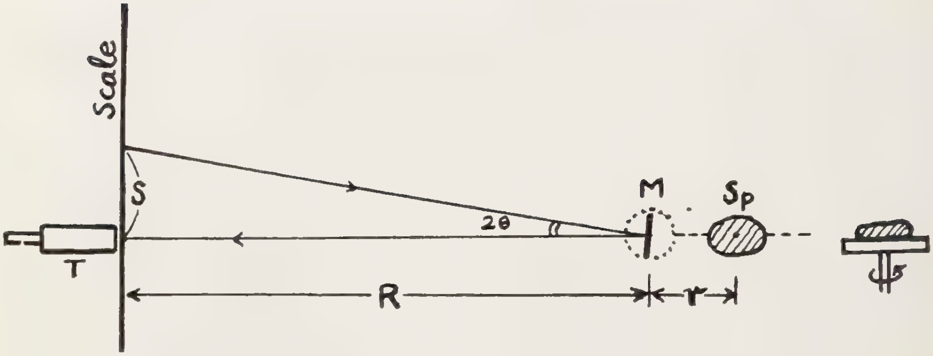


Fig. 15 Schematic representation of experimental arrangement.

with the rotating axis of the holder. The specimen is rotated by every 30°, and at each time the scale and telescope methode is applied.

The magnitude of the magnetism is proportional to *tan* θ, where θ is the deflection of the needle. However, as θ is a small angle, using the formula

$$\theta = \frac{1}{2}\, tan^{-1}\left(\frac{S}{R}\right) \doteqdot \frac{1}{2}\,\frac{S}{R}\,,$$

the magnitude of magnetism in each direction can be supposed to be proportional to S (mm).

After researching the magnetism about a number of Sara-ishi by the methode mentioned above, the author found out that they are classified into two groups. One of them has the magnetism and the other has not it. Generally speaking, Sara-ishi, whose core substance is weathered lava, tuff-like lava or piece of glass, has no magnetism, while the one, whose core substance is other kind of lava than above or volcanic bomb, posseses magnetism.

Observing by microscope, it is found that, in the magnetized core, there always exist the small crystals of magnetite, while, in the weathered lava or tuff-like lava, they could not be recognized, but rather the ferric oxide is observed.

After all, we can conclude that the magnetism of Sara-ishi originates from the magnetism in the core part, and the core-part magnetism is based upon the existence of the magnetite in there.

Furthermore, about the magnetized Sara-ishi, the author read the deflection S (mm) of the astatic needle letting the specimen rotate by every 30°, and analized the results in the following way. Let the maximum and minimum values of S (mm) be a_{max} and a_{min}. Denote each of the observed values as a_1, a_1', a_2, a_2', ······ in the way as shown in Fig. 16 (c). Plot a_{max} and a_{min} on the X-axis in the XY-plane in the suitably chosen scale. Write abscissas at regular intervals and plot a_1 and a_1' on the nearest abscissa to the X-axis, a_2 and a_2' on the next one, and so on as shown in Fig. 16 (d). In ploting these points, the points lying in the same side with respect to the line connecting a_{max} and a_{min} are placed in the same side with respect to the X-axis, and the point on the different side, on the other side of the X-axis. Join these plotted point with solid line. (See Fig.

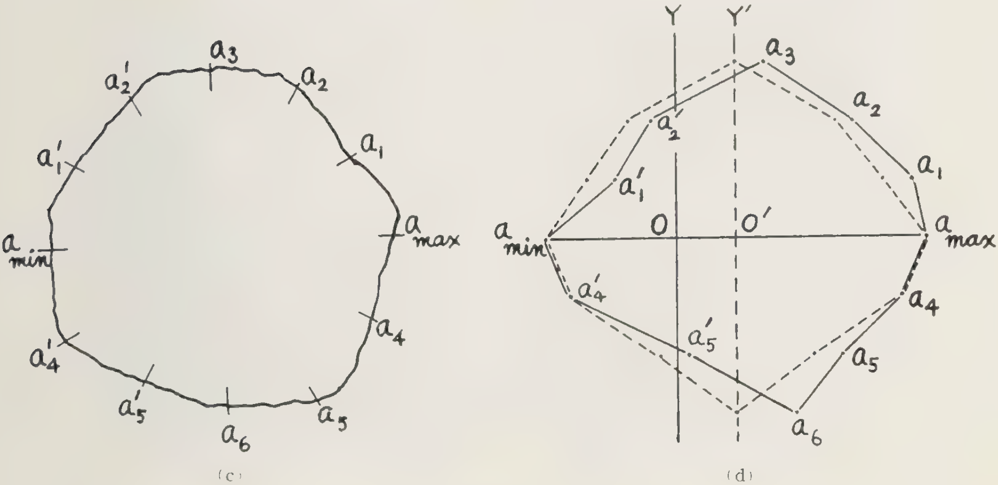


Fig. 16

16 (d)). Draw the line $X=\frac{1}{2}(a_{max}+a_{min})$ and call this line as the Y'-axis. Displace the segments of line $a_1\ a_1'$, $a_2\ a_2'$, $a_3 \cdot a_3'$, ······ so that their middle points may be placed on the Y'-axis, and then, join the edge points of the segments with dotted line. When a_n has no primed counter point, then it should naturally be placed on the Y'-axis. Thus, we can get the diagram symmetrical with respect to the Y'-axis. In relation to this diagram, the previous one might be called "Deformation Diagram" of the distribution of magnetism. Comparing these two diagrams, the author studied about the appearance, especially about the deformation of the actual Sara-ishi. Figs. 17, 18 and 19 show the typical ones. In these figures, (a) shows the upper side of Sara-ishi, (b) its lower side, (c) the measured values of S (mm) at the direction of every 30° and (d) the corresponding diagrams obtained by the way stated above, respectively. Each of these diagrams was examined carefully comparing with the actual Sara-ishi in question.

i. Fig. 17

The upper side of this Sara-ishi is flat and shadow line separates this side into two part, A and B. Part A is low, B is high and the difference is about 3mm. In the downward of Fig. (c) the part enclosed by the dotted line is the marks of hollow in the lower side of Sara-ishi which was caused when the block of lava crashed onto the ground. Therefore, it seems natural to consider that the direction of smashing of this lava is along the arrow line, and when it strikes the ground, its crust part slips to the same direction, which cause the difference between A and B.

On the other hand, in Fig. (d) the upper half is shifted towards the left and lower half towards the right. This is conceivable to indicate the oblique deformation of the distribution of magnetism.

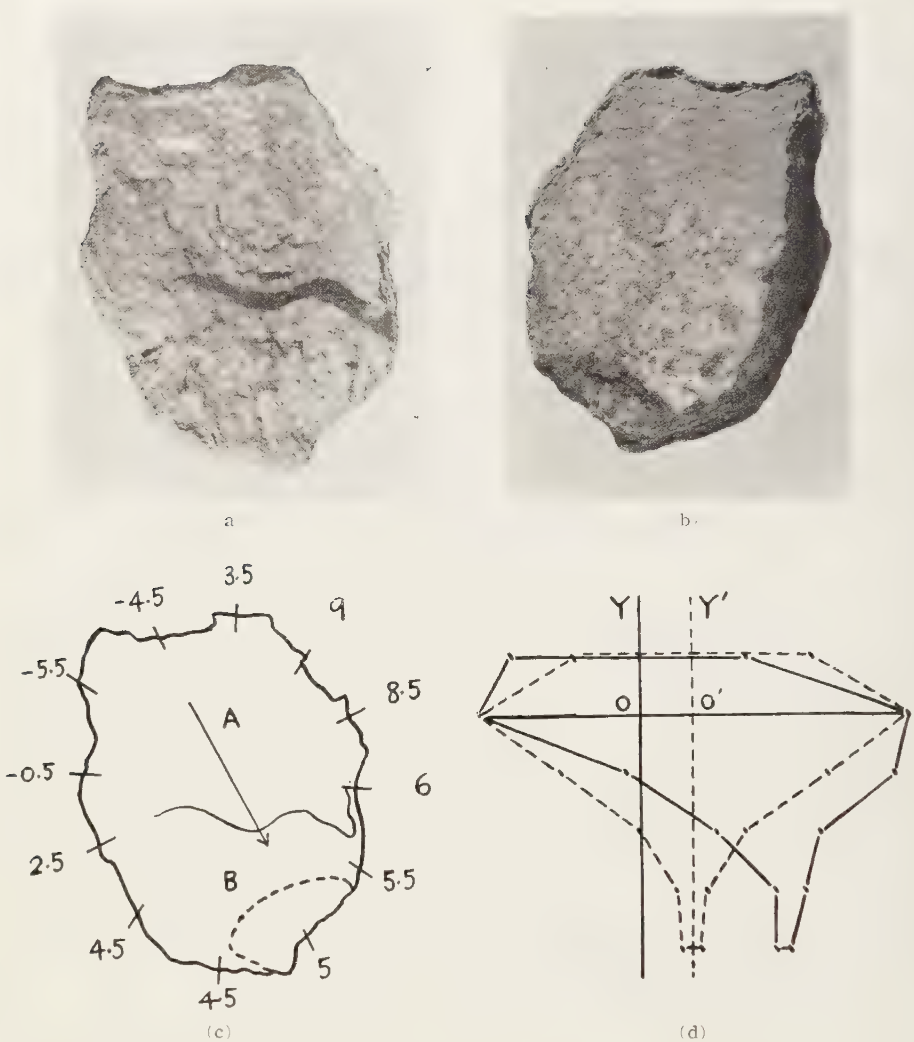


Fig. 17

These two kinds of deformations seems to have an intimate relationship, or rather, as they are in accord with each other quite well, we can say that they come from the same origin. Indeed, we can foresee the distribution of magnetism of Sara-ishi from its external appearance, vice versa. The accordance of the predictions with the observations is satisfactory.

ii. Fig. 18

In Fig. (c) the dotted curve shows the position of the ridge in the lower side of actual Sara-ishi, and from the ridge the thickness gradually decreases. The solid curve

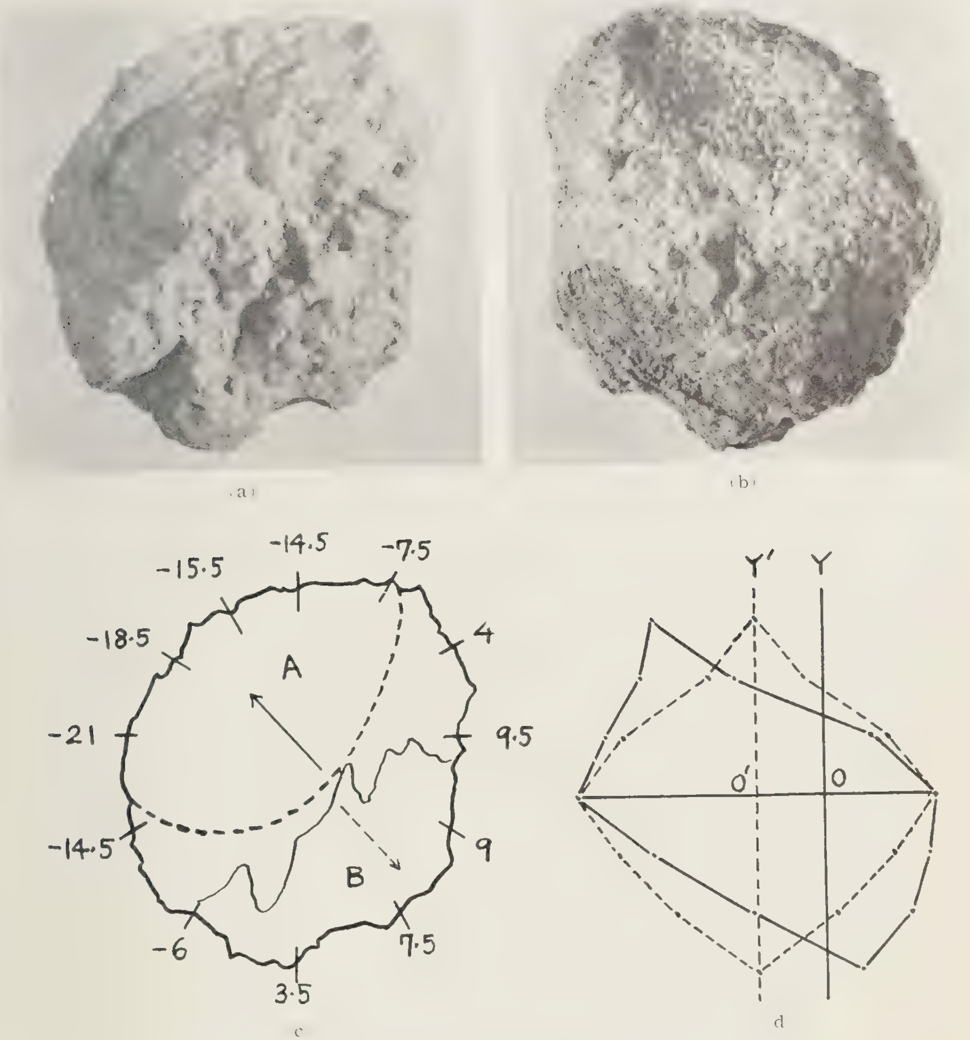
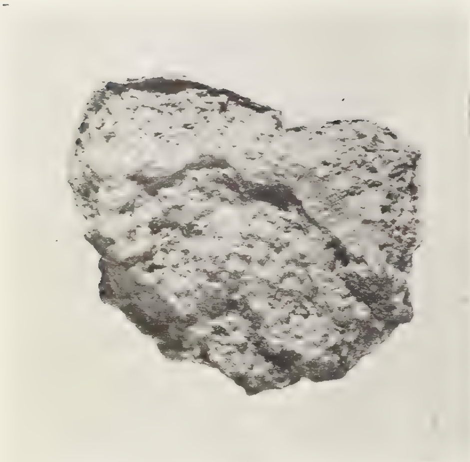


Fig. 18

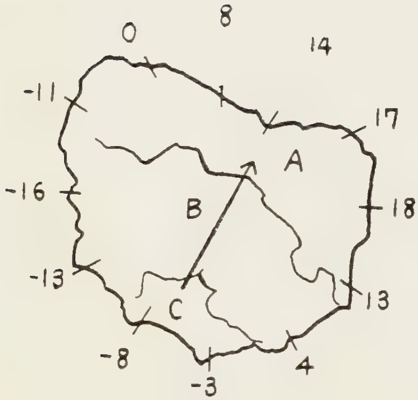
indicates the position of the crack in the upper side of Sara-ishi. It is suggested that, at the crack, the crust part of the lava was torn and at the same time, the parts A and B were displaced along the solid and dotted arrows, respectively. It seems that, when the lava fell, it smashed onto the ground at the inclined surface of the lower side in the direction of solid arrow, and the two part of the crust slipped oppositely to each other.

In Fig. (d) the upper half of solid diagram is shifted to left compared with the dotted one and lower half is shifted to right. Therefore, this diagram indicate the oblique slliping of Sara-ishi as the same as previously stated in the case of Fig. 17.

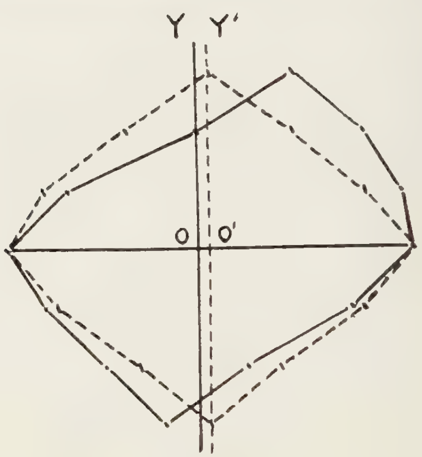


a

b



c



d

Fig. 19

iii. Fig. 19

As is shown in Fig. (c), there are two lines in the upper side by which this side is divided in tiers into three parts A, B and C. This means that the crust part slips along the arrow line and is reconcile with the diagram (d) showing the oblique deformation of the distribution of the magnetism.

So far, the author has exhibited only a few typical examples. The accordance of the deformation diagram and the apparent deformation of Sara-ishi is quite satisfactory for all of the other samples.

After all, the author is led to conclude about the magnetization of Sara-ishi in the

following way. Firstly, the spontaneous magnetism of the magnetite arranged in the block of lava, the main part of the core of Sara-ishi, is in the direction of the earth magnetism and reaches almostly to the saturation state before it reaches to the ground. When the lava strikes the ground, if it deforms, the distribution of the magnetism is also deformed.

In general, when the magnetite of high temperature is cooled in the magnetic field, it is magnetized strongly in the direction of the field when it passes through the Curie point $\theta = 578$ C. Therefore, the temperature of the lava, the core part of Sara-ishi, should be considered to fall down under 570°C before it reaches to the ground.

V. Formation Mechanism of Sara-ishi

As stated in the preceding sections, Sara-ishi is formed in the following way, that is: firstly, the Yona particle is coagulated around the core substance and forms its crust part, and then the rim of it hangs down gradually to make a so-called a dish-like form, at the same time, the little prominences or the stalactitic processes are produced on its inner surface.

There are two theories concerning the formation mechanism of Sara-ishi. One of them is as follows(6). A portion of the volcanic vent was disturbed and shattered by explosive action; and the disrupted rock-fragments thus formed as well as the surfacial volcanic fragmental materials became coated with newly erupted lava mingled with ashes: they were then thrown out and dashed against the surface of the ash-field upon falling. The lava coating on each block was then still in a semi-fluid state so that the upward-projecting rim of the crust was produced by the impact action. According to this theory, Sara-ishi is formed suddenly at the moment of impact.

Another one is that the fine particles of the volcanic ashes, accumulated on the lava-fragment on the ground, are coagulated by the action of the colloidal silicic anhydride and as the result of it, Sara-ishi is formed as the secondary effect(1).

These two theories are essentially different. Now, according to the results of the author's investigation about Sara-ishi, especially about its magnetism, the temperature of the core-part lava should be under 570°C when it reaches to the ground. At this temperature, the lava is considerably hard and it have scarcely any fluidity. Accordingly, it is absolutely impossible to consider that when the lava crashed onto the ground, the marginal part is lifted up suddenly by the impact action so high and sharp as is actualy observed in Sara-ishi.

Further, granting that Sara-ishi is to be a sort of volcanic bomb and is produced primarily from the semi-fluid lava by the impact in the time of volcanic eruption, as the core part should be more fluid than the crust part in the temperature about 570 C, when the marginal part arise, edge of the core part should also arise, or rather, the up-rising of the marginal part is caused by the up-rising of the edge of the core part, and core part should have the form almost similar to the external form of rim as is shown in the dotted line in Fig. 20. This is not the case. In actual Sara-ishi, it is evident from the figure that the shapes of its core part and its crust part are quite independent. Figs. 11 (a) and (b) are their typical examples. Furthermore, we can not understand the difference of the fussing points between the core part and the crust part

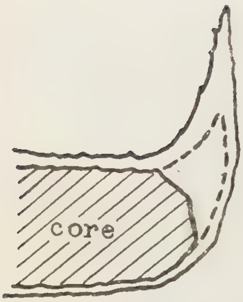


Fig. 20

from the above point of view. After all, it is absolutely unacceptable that the lava suddenly transforms into Sara-ishi by the impact action when it dashes to the ground.

Hereby, the author has stood upon the second point of view that is Sara-ishi is so formed that after the lava fell down on the ground, the Yona is accumulated on its surface and coagulated. As the binder by which the Yona is coagulated on the surface of lava has not been made clear before now, the author has studied it and found out that the binder is the colloidal silicic anhydride and further confirmed it by the following experiments.

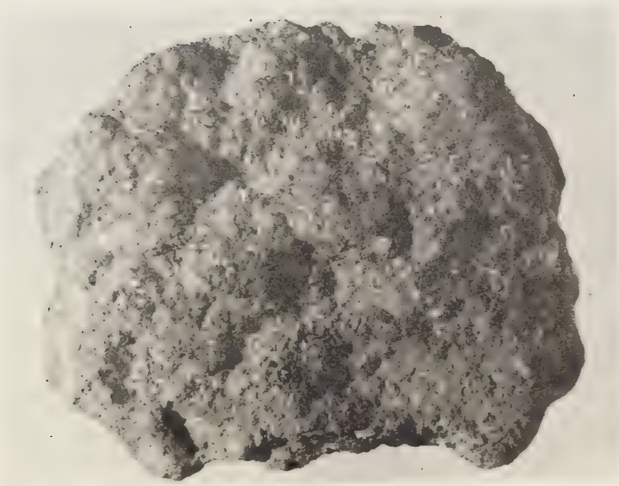
Firstly, cover the bottom of the Schale (dish) with volcanic sands, scattered volcanic ashes and then put the test piece of the lava on it. Then, scatter again the Yona through a sieve upon the surface of the lava to make a thin layer of fine Yona particles on the surface and expose it to the out-door sunlight with wetting it using spray. After it entirely dries up, wet it again and repeat the same procedures. Sometimes, wet it with water containing a little amount of sulphuric acid. Sometimes, after wetting it, put it into the SO_2-atmosphere. After passing twenty or thirty days, wash the surface of the sample lightly and carefully by spraying it with sufficient amount of moisture. Finally, upon the sample thus obtained, scatter again the fine Yona particle through a sieve and repeat the whole procedure again and again for about a year or more. Through these processes, the author has obtained such things as shown in Fig. 21 (a) and (b). Fig. (b) is the original form of the sample. Comparing these two photographs, it is easily found that, during the processes mentioned above, some kind of substances have been coagulated on the lava surface.

Therefore, it is reasonable to consider that, the Yona, after accumulated on the surface of the lava or of the lava-like substances, changes into ash-water under the influence of humidity or rain-water, and when water contained is missing by flowing away along the surface of the lava or by evaporation by, for example, the heat of the sun, the silicic anhydride contained in the Yona changes into the colloidal one by the weathering action and coagulate the Yona to the place where it is.

The ash water flows down along the surfaces of lava or of the likes and then it brought to its edges of the lower side, especially to the rim, and finally hangs there by the surface tension of it. When the water in this part is evaporated, the contents are left behind. In the long time, after such processes being repeated succeedingly, at last, the remnants come to be coagulated and finally form the characteristic hanging rim of Sara-ishi. Naturally, this part of Sara-ishi grows with the positive geotropism and besides, it is clear from the origin of this part, that it contains finer particles of Yona than the upper side of the crust part of Sara-ishi. The state of the layers of flow lines which is found in Fig. 11 gives a good support to the above mentioned interpretation concerning to the growth of the hanging rim of Sara-ishi.

Concerning the growth of the wart-like low prominences and the stalactitic processes on the low side of Sara-ishi, the author consider as follows. When the porous lavas are soaked in the ash-water, the fine particles of Yona permeate into them with water

a



b

Fig. 21

and then percolate through them to the part of the lower side surface. Especially to the part where there is vacant space between that part and the ground, and then they hang in the forms of drops and are dehydrated. Thus, in the end, the contents of the drops are coagulated there and form the low prominences. Accordingly, the formation process for these low prominences is similar to the one for the stalactite in its cave. When the upper side surface of the lava is once covered with the compact thin layer of the coagulated substances, the growth of the prominences is generally stopped, because the ash-water can not permeate into the lava. But, sometimes, when the lava get wet with the ash-water, such as by the rain-fall, drops can be hung down from the low prominences, dehydrated and contents of the drops can be coagulated there again. Thus, some of the low prominences grow to the stalactitic processes. In fact, the stalactitic processes and the low prominences are never found for Sara-ishi having the dence core, but are found only for the one having the core of porous structure.

Sara-ishi is generally considered to be formed in the lapse of 150~300 years. "Maru-shō-hachi", the lava which has been passed for thirty years since it fell down on the spot, shows clearly the early stage of the formation of Sara-ishi, viz., as previously stated, the crust part of its upper side is so thin that the texture of the lava is visible, but its lower side is thicker.

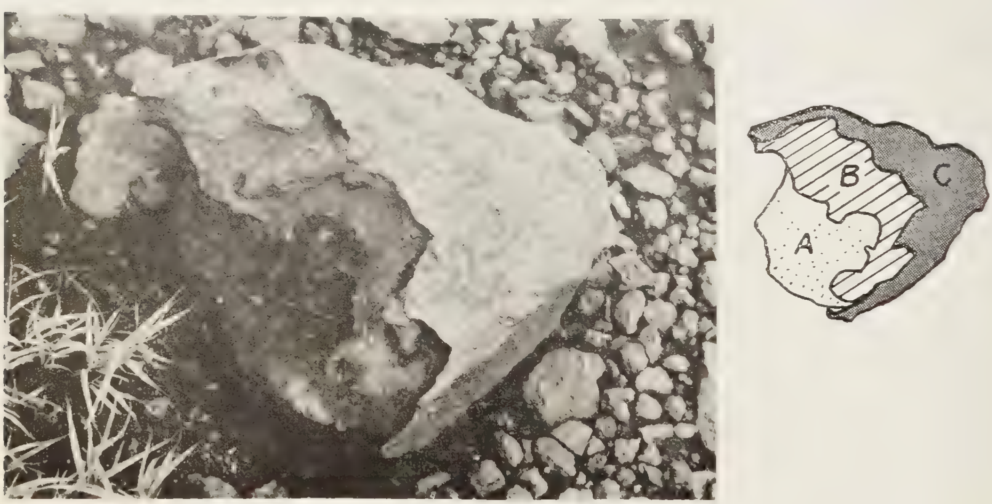
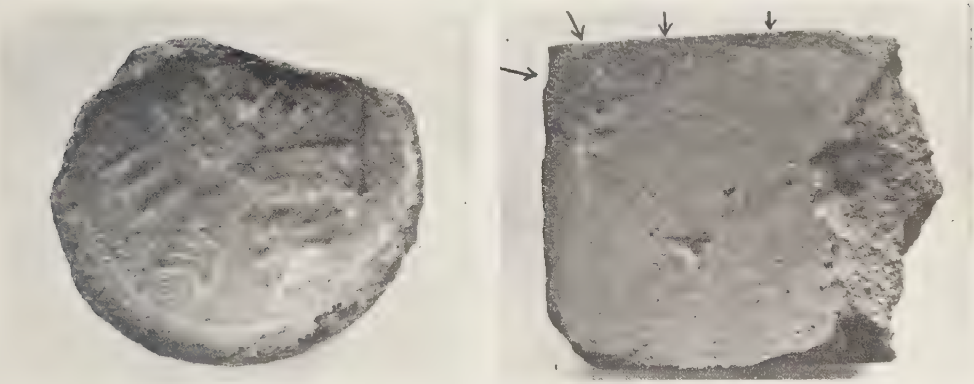


Fig. 22 Sara-ishi which was destroyed two times by some kind of external force.

As the Sara-ishi shown in Fig. 22 has two marks which informs us that it was destroyed two times by some kinds of external forces, one must consider that the state of the developement of it should be divided into three stages. Indeed, it is easily observed that the amount of the coagulated substances are different in each of three parts of its crust part. In the cut-section (part A of Fig. 22), there is only a little amount of the coagulated substances even if there are, and the texture of the core part shows itself in almost all of its section. More amount is in part B. The amount of the



(a) (b)

Fig. 23

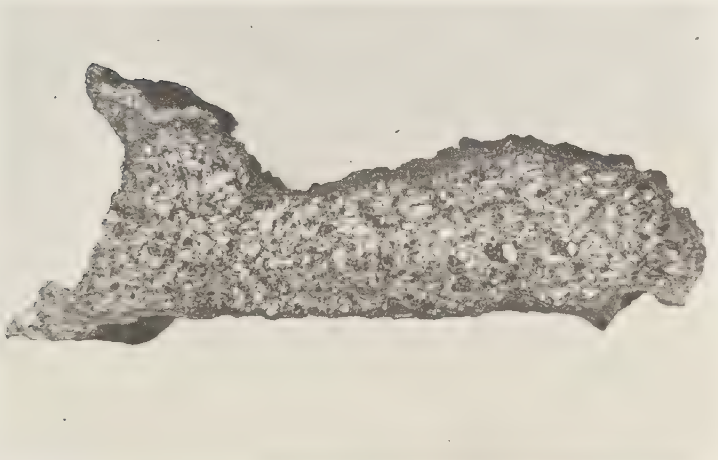


Fig. 24

coagulated substances is the most in the part C. The example of this kind gives a good support to our interpretation concerning to the formation mechanism of the Sara-ishi.

Thus, it is said that the Sara-ishi is so formed that the Yona, the volcanic ashes of Mt. Aso, have accumulated on the lava or other things and then are coagulated. According to this point of view, the Sara-ishi can be formed not only from the lava or such things as its core part but also from the every things lying in the summit area of Mt. Aso, such as every kinds of rocks and the pieces of the glass. The stone ornaments in the shrine compass at the summit of Mt. Aso are the examples.

Fig. 23 shows the other examples which are collected by the author. (a) is the one having the piece of broken glass jar as its core part and (b) is obtained by leaving the piece of brick alone at the top of Sara-yama for nine months. The growth of the coagulated substances in the part of its rim is remarkable. This also gives the strong

support to the author's point of view concerning the formation mechanism of Sara-ishi.

Finally, Sara-ishi which has been turned out in the course of its formation, has the possibility that it has the hanging rims in both of its upper side and of its lower side. Fig. 24 show the very examples of such possibility.

Acknowledgments

The author wishes to express his hearty thanks to Prof. T. Fujiwara, Hiroshima University for his kind guidance and encouragement, and also to Prof. M. Namba for his kind advice and encouragement during the course of the investigation.

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ACOUSTIC ANEMOMETER

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(Received September 30, 1960)

Abstract

An acoustic unidirectional anemometer which indicates absolutely the wind velocity with a rapid response is described. Although the acoustical method is used, the influence of temperature, humidity, density of air and other factors which may change the sound velocity are canceled. Suppose that two sound waves which start at the same time $t=0$, propagate in opposite directions and that, after travelling the same distances l, they reach two microphones at the times $t=t_r$ and $t=t_l$ respectively. Then $t_r=l/(c-u)$ and $t_l=l/(c+u)$ where c is the sound velocity and u the component of the wind velocity along this direction. The difference $1/t_l - 1/t_r = 2u/l$ is proportional to u, and independent of c. If the values of function k/t^2 at $t=t_r$, t_l are observed, we can find the wind velocity. In order to draw the curve k/t^2 by the electronic device, the RC-discharging circuit is used. Of course it draws the curve k/t^2 approximately, we can obtain the true curve as far as we desire.

The device is as follows. The pulse from the pulse generator sounds two speakers which are set back to back. These sounds are received by two microphones which face to the speakers respectively. The initial pulse is also a signal at the beginning of the curve $1/t^2$.

On the other hand two multivibrators which are shifted to one state by the initial pulse and to another state by the received pulse, gate the voltage-time curve described above. The total electric charge that are caused to flows through a known resistance by the difference of the two gated voltages is proportional to the integral of $1/t^2$ from $t=t_r$ to $t=t_l$, which is desired.

This device is capable of measuring the wind velocity from 0 to 40 m/sec. in accuracy within 10 cm/sec. Small improvements will extend the limit of measurement of the sound speed and increase the accuracy to 1 cm/sec..

§ 1. Introduction

Anemometers are divided into three groups. (i) The pressure anemometer; in which the pitot tube serves not only as a standard measure of the anemometer, but as an accurate measure, although, it cannot avoid the influence of the density air. (ii) Using the mechanical movement; For example Robinson's anemometer is the common aparatus because of the minor influence of the air density, and because it able to measure the wind speed of a wide range. However this does not give an absolute measure because of its mechanical structure. (iii) Hot wire anemometer excels in sensibility, rapidness of response and is small, so this kind anemometer is necessary in measureing the turbulent flow. In spite of these qualities, it is neccesary to be corrected when the

air temperature changes. Of course it cannot be standard measure. We will describe the device in which the acoustical pulse waves in the air are used for measureing the wind speed. In brief, using the relations $t_r=l/(c-u)$ and $t_l=l/(c+u)$ we find the unknown u for the observed values of t_r, t_l and l by means of a kind of electronic analogue computer. As the sound speed is canceled, the mesurement is free from the influences of temperature, humidity and other conditions that is to say the influence of the density of air, so it may be regarded as the direct and absolute measurement. The measurable range is 0~40 m/sec. at the present time, with accuracy up to 10 cm/sec., but slight improvements will easily extend the range of sound velodity with accuracy up to 1 cm/sec..

§ 2. Principle

Suppose two sound waves start at the same time $t=0$. In Fig. 1 S shows the sound source at the origin and sounds travel in the opposite directions x and $-x$. Two microphones M_r and M_l which are placed at l and $-l$ respectively. If wind blows uniformly with the velocity V (X-component is u and y-component is v), t_r and t_l are accounted as follows. Consider the two kinds of sound waves; 1. plane waves and 2. spherical waves.

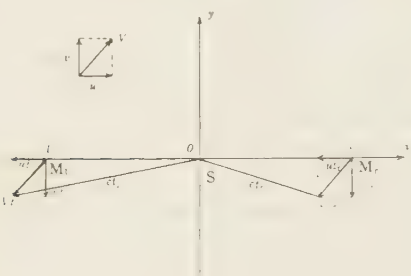

Fig. 1

(i) *Consider the plane sound waves whose front are perpendicular to the x-axis.* In this case we have the relations (see Fig. 1)

$$\left.\begin{aligned} &l-ut_r=ct_r, \quad -l-ut_l=ct_l, \\ &t_r=\frac{l}{c+u}, \quad t_l=\frac{l}{c-u}. \end{aligned}\right\} \qquad (1)$$

Therefore

$$\frac{1}{t_r}-\frac{1}{t_l}=\frac{2}{l}u. \qquad (2)$$

(ii) *Spherical waves.* In this case we obtain (see Fig. 2)

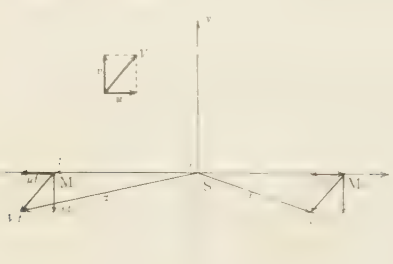

Fig. 2

$$(ct_r)^2=(l-ut_r)^2+(vt_r)^2,$$
$$(ct_l)^2=(l+ut_l)^2+(vt_l)^2,$$
$$\frac{1}{t_r}-\frac{1}{t_l}=\frac{2}{l}u. \qquad (2')$$

Equations (2) and (2′) coincide. Their right sides are proportional to the wind velocity component along the x-axis and independent of either the y-component or the used sound speed. Therefore, if we have a curve which

is proportional to $1/t$ where t is the propagation time, substracting its value at $t=t_l$ from that at $t=t_r$, then we obtain the value proportional to the wind velocity. In practice we use the voltage curve proportional to $1/t^2$ instead of $1/t$, and integrate from $t=t_r$ to $t=t_l$, since it is easier in treatment and its error is less than the former.

§ 3. Voltage-time curve proportional to $1/t^2$

There are various ways to generate the voltage-time curve, but most of them are complicated, however, we obtain satisfactory approximate curve with RC–circuit as follows.

As it shown in § 2, in place of

$$u(t)=-\frac{l}{2}\frac{1}{t},$$

we will take its differential function

$$U(t)=\frac{l}{2}\frac{d}{dt}u(t)=\frac{l}{2}\frac{1}{t^2}$$

and expand it in Taylor's series about $t=t_0$

$$U(t)=\frac{l}{2}\frac{1}{t_0^2}\left\{1+2\left(\frac{t_0-t}{t_0}\right)+3\left(\frac{t_0-t}{t_0}\right)^2+4\left(\frac{t_0-t}{t_0}\right)^3+R_n\right\}, \quad \cdots\cdots\cdots\cdots\cdots\cdots (3)$$

where

$$R_n=\frac{l}{2}\cdot 5\,\frac{1}{\xi^6}(t_0-t)^4. \quad \cdots\cdots\cdots\cdots\cdots\cdots (4)$$

One of the most simple and stable method to supply the voltage-time curve is to use voltage of condenser discharging through a resistance. The curve is expressed as an exponential function or a monotounus function. As $1/t^2$ is a monotounus function too, it will approximate the discharging voltage curve. In a curcuit consisting of a condenser and a resistance, the voltage across the condenser is given by $E_\alpha exp(-\alpha t)$. It is expanded as

$$E_\alpha exp(-\alpha t)=E_\alpha exp(-\alpha t_0)\left\{1+\alpha(t_0-t)+\frac{1}{2}\alpha^2(t_0-t)^2+\frac{1}{6}\alpha^3(t_0-t)^3+\cdots\right\}. \quad \cdots (5)$$

As the left side of Eq. (5) has a pair of arbitrary constants, the coefficients of the first and second terms can be equal to each of which of the ones of Eq. (3) respectively. If we adopt two circuits consist a condenser and a resistance, we can identify four terms, and in the general case of n-pairs, $2n$ terms. It is sufficient, in practice, to adopt tow pairs $E_\alpha exp(-\alpha t)$ and $E_\beta exp(-\beta t)$. In expanding the sum of their voltage we obtain (see Fig. 3)

$$\left.\begin{aligned}E_\alpha exp(-\alpha t)+E_\beta exp(-\beta t)=E_\alpha exp(-\alpha t_0)\Big\{1+\alpha(t_0-t)+\frac{1}{2}\alpha^2(t_0-t)^2\\ +\frac{1}{6}\alpha^3(t_0-t)^3+\cdots\Big\}+E_\beta exp(-\beta t_0)\Big\{1+\beta(t_0-t)+\frac{1}{2}\beta^2(t_0-t)^2\\ +\frac{1}{6}\beta^3(t_0-t)^3+\cdots\Big\}.\end{aligned}\right\} \cdots (6)$$

Comparing the coefficients of the corresponding terms of Eqs. (3) and (6), we get

$$\left.\begin{aligned} U(t) = \frac{1}{4}\frac{l}{t}\Big[&(3-\sqrt{3})exp\left\{(3+\sqrt{3})(1-\frac{t}{t_0})\right\} \\ &+(3+\sqrt{3})exp\left\{(3-\sqrt{3})\times(1-\frac{t}{t_0})\right\}\Big]. \end{aligned}\right\} \quad (7)$$

The suitable circuit of (7) is illustrated in Fig. 3 where

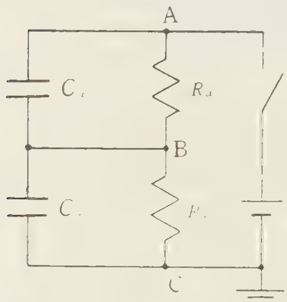

Fig. 3

$$\left.\begin{aligned} R_\alpha C_\alpha = 1/\alpha \\ R_\beta C_\beta = 1/\beta \end{aligned}\right\} \quad (8)$$

The initial conditions are as follows

$$\left.\begin{aligned} E_A - E_B = E_\alpha \\ E_B - E_C = E_\beta \end{aligned}\right\} \text{ at } t=0 \quad (9)$$

where E_A, E_B and E_C are the voltages at A, B and C respectively.

As we have expected, the potential E_A is proportional to $1/t^2$ for all time t, and so if the ratio of the potentials

$$\frac{E_A - E_B}{E_B - E_C} = \frac{E_\alpha}{E_\beta} \quad (10)$$

is equal to the ratio of resistances

$$E_\alpha / E_\beta = R_\alpha / R_\beta, \quad (11)$$

then the capacitances are

$$C_\alpha = C_\beta = \frac{1}{\alpha R_\alpha} = \frac{1}{\beta R_\beta}. \quad (12)$$

Substituting the numerical values $l=5.1\,m$, $t_0 = l/c_0$, $c_0 = 340\ m/sec.$ putting $C_\alpha = C_\beta = 3\mu F$, we obtain from Eq. (4)

$$\left.\begin{aligned} \alpha &= 715.5\ sec^{-1} \\ \beta &= 484.5\ sec^{-1} \\ R_\alpha &= 1.054\times10^3\Omega \\ R_\beta &= 3.944\times10^3\Omega \end{aligned}\right\} \quad (13)$$

Let E_0 be the initial voltage at point A, then the voltage at $t=t_0$ is

$$E_A = 0.221\ E_0 \qquad (\text{at } t=t_0). \quad (14)$$

§4. Appratus

Fig. 4 illustates the block diagram of this equipment. The pulse generated by two vacuum tubesystem is applied to the four grid of tubes 12AU7. The amplified pulse has four functions i. e. sounding speakers, beginning to draw the standard voltage time $1/t$

curve and openning two gates. Next the vacuum-tube acts as a monovibrator by which the width of the pulse is adjustadle and the two tubes 6AS7–G's act as a driver, to which two speakers are connected. When the signal pulse is applied to the third tube 12AU7, monovibrator shifts the plate potential to the excited state and reverts automatically to the quienscent condition in 20 milli-seconds. During the excited state the grid of the 6V6–GT is hold at a low potential low enough that the tube is cutoff. The condensers inserted in the cathode circuit continue discharge so that the voltage across the condensers change proportionally to $1/t^2$. The sound waves from the two speakers propagate in the air and reach the microphones at times t_r and t_l respectively. The received pulses are amplified by the pre– and main-amplifiers, then they are applied to the circuits and make each gate pulsate. The gates which have been opened by the initial pulse are shut by the pulse from the microphone. On the other hand the voltage-time curve is applied across the grid of the tube 12AU7. The cathode potentials of the two diodes 6AL5's inserted in the cathode circuits of the tubes 12AU7 are compared and their difference is displayed on a meter. And the double delay pulse generator is provided for testing. The electric power of all the circuits described above is supplied by a stabilized voltage source.

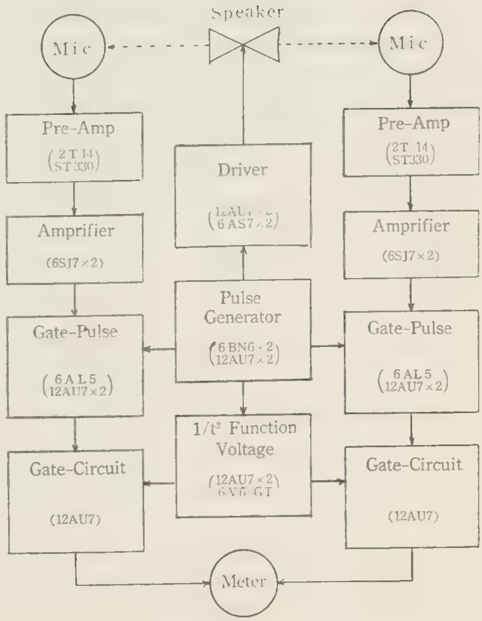

Fig. 4

(i) *Pulse generator and driver*

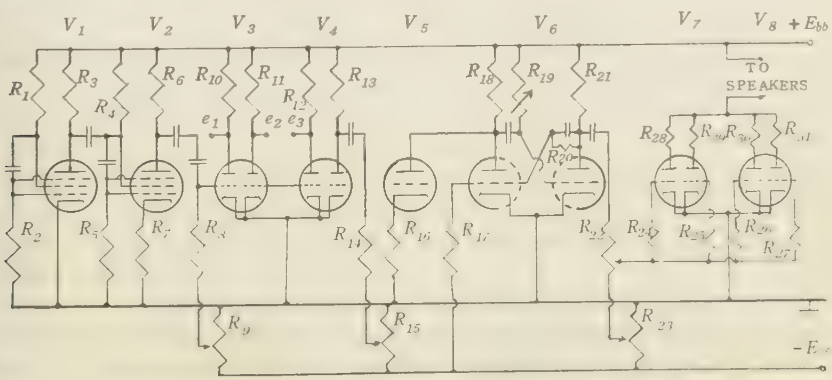

Fig. 5

Fig. 5 shows the circuits of the pulse generator and driver. The negative pulses which are generated in two vacuum-tubes V_1, V_2 (6BN6×2) (30 cycles per second), are amplified, reversed to positive pulse and applied to the four grids of tubes V_3, V_4 (12AU7×2). One of their outputs is connected through a tube V_5 (1/2 12AU7) to a tube V_6 (12AU7) which acts as a monovibrator. By the variable resistance R_{19} the width of the output pulse is adjustable (it is chosen about 50 micro-seconds). The tubes V_7 and V_8 (6AS7-G×2) drive the two speakers. The plate current flows directly through the moving coils of the speakers without output transformers. These speakers are the horn type tweeters (Pioner PT-2) which are inferior in its frequency characteristics above 20 kilo-cycles.

(ii) *Voltage-time curve proportional to* $1/t^2$

The second output of V_6 in Fig. 5 acts as a triggering pulse of the monovibrator V_{10} (12AU7) through V_9 (1/2 12AU7). See Fig. 6. The width of the pulse is adjusted to

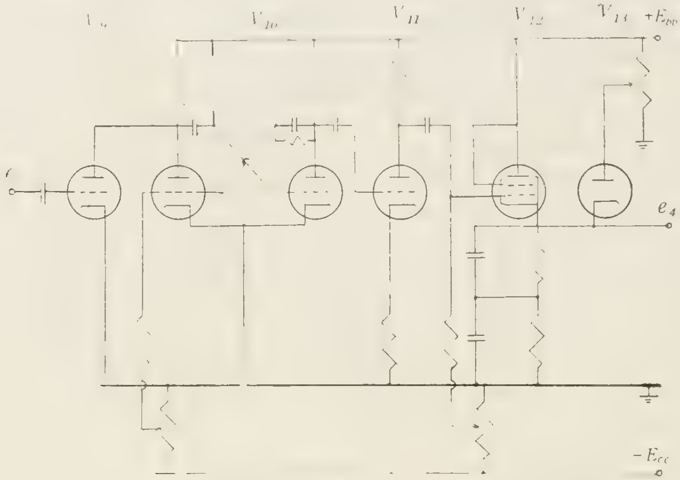

Fig. 6

about 2.0 milli-seconds. The square wave voltage is applied across the grid of V_{12} (6V6-GT) through V_{11} (1/2 12AU7) so that under the quiescent condition V_{11} rest in a cutoff. Because the grid potential of V_{12} is so high that its plate current of V_{12} is saturated, a constant current flows through the resistane inserted in the cathode circuit. As the result the cathode stays at a constant potential. When the signal pulse is applied, V_{11} is driven on, while V_{12} is cut off and the condenser in the cathode circuit begin to discharge through the resistances. The potential of the cathode gives curve proportional to $1/t^2$.

When the monovibrator returns to its initial condition, or V_{11} is cut off, the plate current of the tube V_{12} begins to flow and V_{12} reverts to a quienscet condition. Fig. 7-*a*, *b*, *c* show the initial pulse, the plate voltage of V_{12}, and the voltage of cathode or output respectively. V_{13} (1/2 6AL5) makes the starting volage equal at each cycle. The

cathode voltage E_0 is adjusted to 150 volts when the V_{12} is set on.

(iii) *Amplifier and square wave for gate circuit*

Common audio frequency ampifier are used, because they are comparatively simple and sufficent for our purpose. The pulse received with a crystal microphone is amplified by the pre-amplifier which consists of two transistors V_{14} (low noise 2T14) and V_{15} (ST-300) (see Fig. 8). The switch which is set after the main-amplifier V_{16} and V_{17} (6SJ7×2) is set to the positive or negative side according to the polarity of the received first wave form. If it is positive, after being reversed through the tube V_{18} and if it is negative, without passing V_{18}, the pulse is applied to the plate of V_{20} with a negative pulse regardless of the polarity of the received pulse. The double triode V_{20}, V_{21} (12AU7) acts as a multivibrator, and the initial positive pulse is applied to the plate of V_{21}, so that the tube V_{21} is in the condition in which V_{22} is cut off. When the received negative pulse is applied to

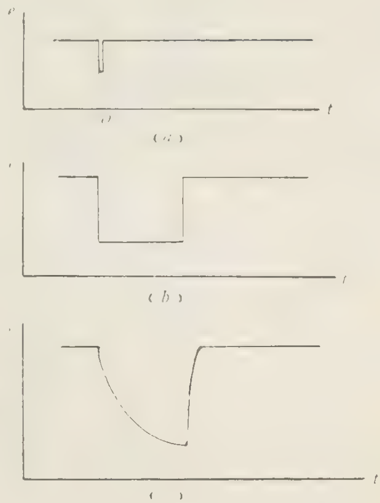

Fig. 7

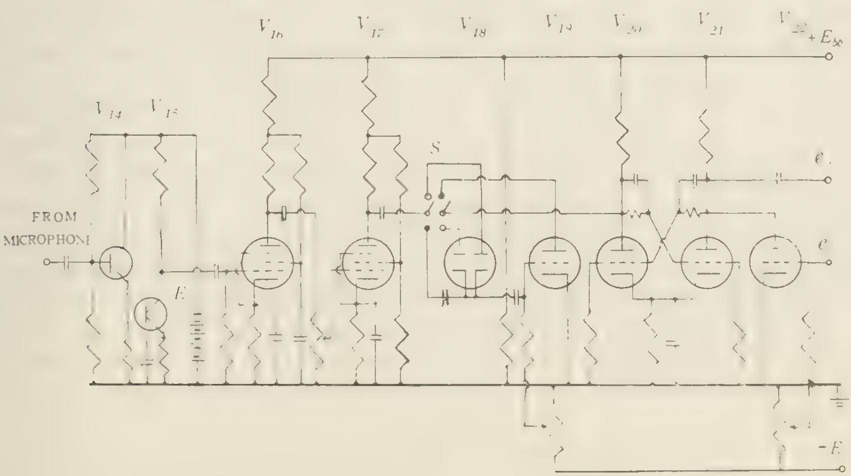

Fig. 8

V_{20} in such a condition the multivibrator is reversed so that V_{20} is on and V_{21} is off. Even though the received wave form is comprehensive, it will be rectified to one polarity and according to the characteristics of the mutivibrator it is sensitive to no waves except the first one. The output at the plate of V_{21} is a square wave which raises with the initial pulse and drops at the time when the microphone receives the

sound pulse. The two circutis described above are used to observe the transmission time of each side respectively.

(iv) *Gate circuit and meter*

Fig. 9 shows circuit of gate and meter. The voltage proportional to $1/t^2$ is applied across the grid of V_{23} and V_{23}'. Before the initial pulse comes, the grid potentials of V_{26} and V_{26}' (6AL5) are below the ground potential and their cathode potentials are equal to the ground potential. After the initial pulse come, the grid voltage drops and V_{26} is cut off. And the cathode voltage of V_{25} becomes

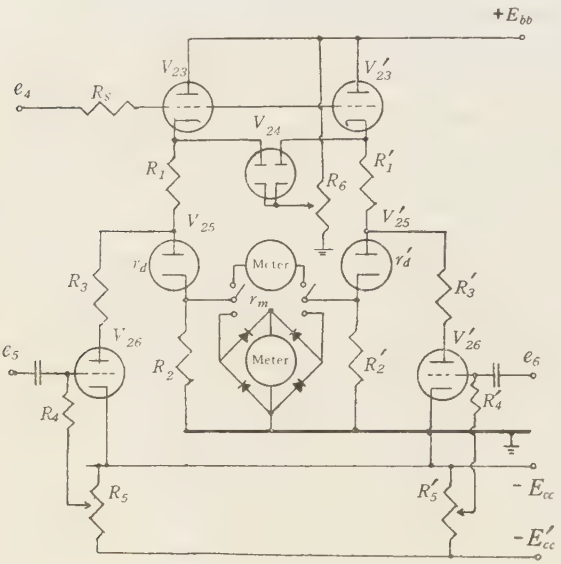

Fig. 9

$$E = E' \cdot \frac{R_2}{R_1+R_2+r_d} \quad \cdots\cdots \quad (15)$$

where E' is the cathode voltage of V_{23} and r_d the resistance of the diode. As the circuit is regarded as a cathode follower and the grid leak R_s is high enough to be disregarded the grid current and E is proportional to the initial standard voltage $1/t^2$, which is equal to the potential of the cathode of the diode V_{25}. Provided that the two circuits are symmetry i. e. $R_1=R_1'$, $R_2=R_2'$ and $r_d=r_d'$ and when the each cathode voltage are equal, no current flows through the resistance (or micro-anmeter) which is inserted in them. Supporse either (say right hand) of microphones has received the signal pulse, and the cathode potential of the diode in this side drops to zero, while the other (left hand) microphone doesn't and the cathode potential is still high. Then some current flows through the resistance which connects the two cathodes in the condition, where either of the microphones has received the pulse and the other has not. Otherwise no current flows. When r_m denotes the resistance connecting the two cathodes, the current cathodes, the current

$$I = \frac{R_2}{r_m+R_2'+R_2} \frac{E}{(r_d+R_1)+R_2/\!/(r_m+R_2')}, \quad \cdots\cdots\cdots\cdots \quad (16)$$

or

$$I' = \frac{R_2'}{r_m+R_2+R_2'} \frac{E}{(r_d'+R_1')+R_2'/\!/(r_m+R_2)} \quad \cdots\cdots\cdots\cdots \quad (17)$$

is proportional to E and continues to flow from the time when either of microphones receives the pulse to the time when the other microphone also recieves it. Integrating

this current, we get the value proportional to the wind velocity. (See Fig. 10)

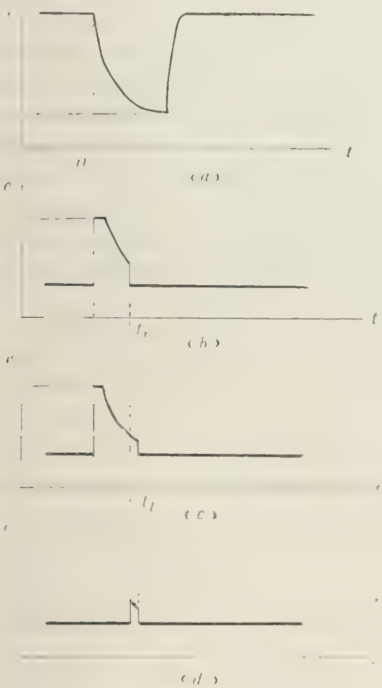

Fig. 10

In practice, $R_1=R_1'=R_2=R_2'=R=10\ k\Omega$, the resistances r_d, r_d' are neglected, and the plate resistance r_m is 10 $k\Omega$. Hence

$$\left.\begin{aligned} I&=E/3R \\ -I'&=E/3R \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (17)$$

Provided that the grid potential of V_{23} and V_{23}' are equal to their cathode voltage, at $t=t_0=l/c$ the current in Eq. (17) is as following:

$$I=0.221\ E_0/30\ mA\ (\text{at } t=0) \quad \cdots\cdots\cdots (18)$$

(v) *Double delay circuit for test*

In addition to the above equipment, the delay pulse circuits for testing and adjustment are presented. A square wave which appears at the output of the monovibrator is differentiated and get a delayed pulse. The delayed pulse is applied to two valves of monovibrators and double delayed pulses are obtained at their outputs. These delay times which are taken as the propagation time of the sound are able to change by variable resistances of the circuits.

§ 5. Test

A test was carried out in a calm room. Sound waves are sent from a source in one direction and they are received by tow microphones one of which is placed at a distance of $l=5.1$ metres from the source and the other of which is at a distance of $l+\Delta l=5.1+\Delta l$ metres. In this arrangement the current through the resistance (micro-anmeter) is

$$I=k\left(\frac{c}{l}-\frac{c}{l+\Delta l}\right), \quad \cdots\cdots\cdots\cdots (19)$$

where c the sound speed is nearly equal to 340 metre per second and k is the constant which is determind by the elements of the circuit. In this circuit they are chosen as $t=t_0=l/c=5.1/340=0.015$ second and $E=0.221\ E_0$, where E_0 is a standared voltage E at $t=0$.

Under this condition, the culculated values and the observed values of the current through the meter versus Δl and the corresponding values of the wind speed are illustrated in Table 1.

Table 1.

$l+\Delta l$	u	I cal	I obs
Metre	m/sec.	μA	μA
5.00	-3.40	-9.95	
5.02	-2.71	-7.93	
5.04	-2.02	-5.91	
5.06	-1.34	-3.92	
5.08	-0.67	-1.96	
5.10	0.00	0.00	0.0
5.12	0.66	[illegible]	[illegible]
5.14	1.32	3.86	3.9
[illegible]	[illegible]	[illegible]	[illegible]
5.18	[illegible]	[illegible]	[illegible]
5.20	[illegible]	[illegible]	[illegible]
5.22	[illegible]	[illegible]	[illegible]
5.24	4.59	13.42	13.3
5.26	5.17	15.10	15.0
5.28	5.80	16.95	17.0
5.30	6.42	18.76	18.8
5.32	7.03	20.54	20.5
5.34	7.64	22.33	23.0
5.36	8.25	24.11	24.5
5.38	8.85	25.87	26.7
5.40	9.45	27.62	28.6
[illegible]	10.04	[illegible]	
[illegible]	[illegible]	[illegible]	
5.48	11.21	32.77	
5.50	12.36	34.47	

§ 6. Discussion

To estimate the error, the following condition is considered as a standard case; the distances from the sound source to the both microphones are equal i.e. $l=5.10$ and the sound speed c is 340 m/sec. As a result the propagation time in calm air is $t=l/c=0.015$ micro-second. If the wind blows at a speed of u m/sec., the difference of propagation time is, neglecting higher order terms than the second of u/c,

$$t=\frac{l}{c-u}-\frac{l}{c+u}\doteqdot\frac{2l}{c^2}u=88u\times10^{-6}sec.. \quad (20)$$

That is to say, the first approximation of the time difference is about 88 micro-seconds by the wind at a speed of 1 m/sec..

(i) *The error arised from the distances between speakers and microphones*

If a microphone is set at a distance of l metre from the source and another at $l+\Delta l$ m, the error Δt is expressed as $\Delta l/c=0.0029\cdot\Delta l$ second in terms of propagation time difference, or in terms of wind speed the error Δu is expressed as

$$\Delta u=\frac{\Delta l}{c}\cdot\frac{c^2}{2l}=33\cdot\Delta l\ m/sec. \quad (21)$$

It means that if a microphone is placed at a distance of Δl metre from the correct position, by the observed value we obtain $33\cdot\Delta l$ m/sec. which is greater than the true value. And if it is desired to observe the wind speed with an accuracy of 10 cm/sec., the distance is to be measured accurately to 0.1/33m=0.3cm.

(ii) *Acoustical and acoustic-electrical transformation errors*

(a) The error by the propagation in air

We will consider the rise time of pulse for we use the initial pulse only. The rise time of a acoustical square wave cannot be increased more than one micro-second by the propagation in air in our case.

(b) The acoustic-electrical transformation error

The mechanical time lag due to the mechanical inertias of the speakers and the microphones is less than a few micro-seconds. Furthermore as it works in the same way on both sides, their difference becomes less than a micro-second.

(iii) *The error in the electrical circuits*

(1) The mono- and multi-vibrators in the circuit are set in a few micro-seconds from

each initail condition to another condition and as the above (ii)-(a), they works the same way on both side and so we can neglect their time lags.
Several valves are used, but the small change in either their plate resistance or amplification affect the resulted value as error. The constants of the circuit elements for standard curve is most effectable.

Now consider the error which arise by the adoption of a nearly equal curve instead of the true curve $1/t^2$. In the present case, by two pairs of a resistance and a condenser, the four coefficients of terms from zero order to the third order are identified and the error is less than the fourth order. For example, suppose that the maximum velocity is 60 m/sec. and the error will be $(60/280)^4=3.1\times10^{-3}$. If the measurement at a higher velocity or the more accurate measurement is desired, it is most effective to increase circuit components which consist of a resistance and a capacitance but at such a high speed as $u/(c-u)$ approaches to unit, the convergence of the series becomes worse.

The errors in each circuit element are as follows: The capacity C_α affects the time constant α while the resistance R_α affect the time constant α and the initial voltage E_0. If the time constant α has varied from α to $\alpha+\Delta\alpha$, then the voltage $E_\alpha exp(-\alpha t)$ changes at the rate $\Delta\alpha/\alpha$, and $E_\beta exp(-\beta t)$ changes at the same rate. The variation of the initial volgate, $E_\alpha\rightarrow E_\alpha+\Delta E_\alpha$ is hold in the same rate. For the sake of accuracy within 1%, the consitants of elements are to be accurate within 0.3%.

(iv) *The current flowing the gate circuit expressed in Eqs.* (15) *and* (15′).

As $R_1=R_1'=R_2=R_2'=10\ k\Omega$, the plate resistances r_d of diodes may be regrded as $r_d\ll R_1$ and $r_d'\ll R_d'$.

As the other resistancs are constant, the standard voltage is adjustable within the accuracy of 0.1%. The cathode voltage is expressed as follows:

$$\left.\begin{aligned} &E-AE \\ &A=\frac{\mu Z}{r_p+(1+\mu)Z} \end{aligned}\right\} \qquad \cdots\cdots (22)$$

where

E=grid voltage (equal to standard voltage)
A=voltage amplification
r_d=plate resistance=7.7 $k\Omega$ (in 12AU7)
μ=amplification factor =17 (in 12AU7)
Z=cathode impedance=20 $k\Omega$.

Therefore $A=0.925$ and the variation of A with μ is of the higher order than the 2nd order.

(v) *Variation in the supply voltage*

The equipment has a good statility for the variation of the supply voltage. The variation of the supply voltage affects only the frequency of pulses and the voltage in

the standard voltage curve. And so the stabilization of the accuracy within 0.1% is satisfactory.

§7. Summary

This equipment may be the absolute measurement of the wind speed by means of the acoustical method. Even though the acoustical speed may change by any factor, no effect occurs at the observed values and it involves no mechanical inertia. The wind speed may be display directly on the micro-anmeter to the accuracy of 0.1 metre per second. Its defects as follows. As the Robinson's anemometer shows the integrated value of the wind speed with time, this equipment shows the integrated value of the wind speed with the distance from the speaker to the microphone. Furthermore as we take the difference of two paths, the irregularity of the air condition affects the results considerable. Shortening of the distance may lead to some improvement, but we must take account of the disturbance on the wind by the speakers and microphones. Moreover it is suffered by the acoustic noise. For example if an outer acoustic noise acts on the one microphone soon after the initial pulse, very large current flows through the meter so it shows a very high wind speed, (The Kyushu University where the auther performed the experiment is located so near Itazuke Air Port that the observation was often prevented by the leaving and approaching of airplanes.). To avoide this difficulity, several methods are suggested: 1. to design a shorter distance between microphones and speakers, 2. increase the loudness of the sound source, and 3. to decrease the sensitivity of microphone etc., but the essential problem still remains.

Our method may be applied not only to the measurement of the wind speed, but also the measurement of the flowing speed of other gases or liquids.

Acknowledgement

The author wishes to express his gratitude to Prof. Miduno for his useful suggestions and encouragement and the late Prof. Okazaki for his encouragement. The author also wishes to thank Assc. Prof. Matumura and the other members of Department of Physics, Faculty of Science Kyushu University for their kindness in offering facilities for his use.

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PAPER CHROMATOGRAPHY OF INORGANIC IONS BY USING ORGANIC ANALYTICAL REAGENTS IX

Precipitation Chromatography of Cations with 8-Quinolinol (Part 7)

Hideo NAGAI

(Received September 30, 1960)

Aiming at the use of the paper precipitation chromatography of inorganic cations with 8-quinolinol (oxine) as a simplified method of quantitative analysis, the author has previously reported on the correlation between the quantity of cation and the zone area or the zone width in regard to copper and iron. In order to obtain the fundamental data, the experiment was performed under so restricted conditions that the procedure was too tedious to be brought into practice as a simplified quantitative method. In the present study, the author intended to accomplish a practically simplified method of quantitative analysis by improving the too strong affectability to temperature change and the rather poor applicability to concentration of the test solution, as well as the too tedious operations. The affectability was improved by using isobutanol instead of *n*-butanol to saturate a test solution or a developing solvent. Thus, the results were almost the same under the operations in the temperature range of about 10°~20° C. The applicability in concentration of the test solution was somewhat improved by regulating the *p*H of the test solution and the developing solvent.

Introduction

When the fundamental study of paper precipitation chromatography for a simplified method of quantitative analysis was performed , the most notable defect was that it was too strongly influenced by temperature change. Therefore, to obtain a satisfactory result, the thermostat was indispensable but it would be preferable to do without it to a simplified procedure. The affectability to temperature change was probably caused by the use of *n*-butanol which was added to saturation to a test solution or a developing solvent to make the developing smooth. However, the amount of *n*-butanol reserved in the water phase decreases according to the temperature increases 8.91 % at 10, 8.21 % at 15°, 7.81 % at 20° C[2]). Finally, when the development was made at a higher temperature than 20° C, the amount of *n*-butanol was frequently insufficient for the development even for the qualitative purpose . Therefore, isobutanol which is analogous to *n*-butanol in its chemical nature but more soluble in water (9.8 % at 10°, 8.5 % at 20° C[2])) was adopted instead of the *n*-butanol. The chromatograms obtained by the use of the isobutanol were almost the same under the temperature range of about 10~20 C. The second defect of the chromatography was the rather narrow concentration range of the test solution available for the quantitative purpose. This defect was caused by the poor separability between the copper and iron zones, and the non-homogeneous distribution of

the precipitation of the oxinate in each cation zone. To improve these defects the test solution and the developing solvent were regulated within or near the *p*H range where the metal oxinates were stable. However, the development at a quite stable *p*H for oxinates was apt to cause too abrupt and abundant precipitation around the center of the filter paper, resulting in failure of the smooth development. On the other hand, the result of the previous experiment suggested that the linear relation between the quantity of cation and the zone width might be more widely applicable than that between the quantity and the zone area would be. These phenomena were probably caused by the development at a *p*H which is rather unstable for the metal oxinate. In such a case the distribution of the precipitation of metal oxinate was not quite homogeneous: it showed changes according as the distance from the center of the filter paper. The development in this study was, as described above, tried in a *p*H range where the metal oxinates were almost or completely stable, so that the distribution of the metal oxinates was almost homogeneous in every portion of the zones. Accordingly, the concentration of metal cation will be more closely related to the zone area than to the zone width.

Experimental

The experiment was made in almost the same way as was reported before[1] with the few exceptions which are as follows:

1) The treatment of the filter paper with oxine did not need to be so precise as was reported before[1]. The following was enough to secure a satisfactory result: the filter paper treated as was reported before[3] was stored in a closed vessel and used immediately after heating about an hour in an electric drier at $80^\circ\sim 90^\circ$ C.

2) The test solution was buffered with the addition of the equivalent volume of the hydrochloric acid-sodium acetate buffer solution of *p*H 5.4[1], then saturated with isobutanol, and applied to the filter paper impregnated with oxine; the capillary dropping tube containing 5 μl test solution in it was made to touch slightly and perpendicularly on the central portion of the filter paper impregnated with oxine. Then the filter paper was rotated around the capillary tube. With these operations the test solution gave spot of almost perfect circle on the paper. The developing solvent used was the hydrochloric acid-sodium acetate buffer of *p*H 3.1[4] saturated with isobutanol.

3) The development reported in the previous paper[1] was made in a rather unstable *p*H value for the oxinates, so that the exposure to ammonia vapour was necessary at the end of the development to stabilize the precipitation of the oxinates. On the other hand, the development in this study was done at an almost stable *p*H value for the oxinates, so the procedure of exposing to ammonia vapour could be eliminated and the chromatogram was immediately dried on an electric heater so as to fix the zones as soon as possible. The development of the color of the zones was sufficient for the identification of the metals. With these procedures, the cupric and ferric oxinates were precipitated in concentric circles and the distribution of the precipitation was almost homogeneous in every portion of each zone (copper appeared at the central part in yellow coloration, and iron followed around it in black coloration). Fig. 1 shows the correlation between the quantity of iron and the area of the zone, and Fig. 2 shows the

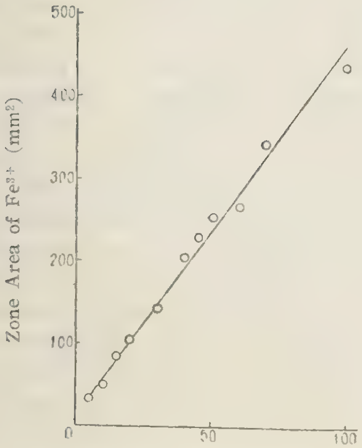

Fig. 1

Fig. 2

correlation between the quantity of copper and the area of the zone. In these cases each of the zone areas was not influenced by the coexistence of the opposite cation ranging 5γ/5μl～20γ/5μl. Therefore, it may be reasonable to say that the correlation between the quantity of the cation and the area of the zone is practically linear in 5γ/5μl～100γ 5μl.

Discussion

1) When the test solution saturated with *n*-butanol was applied without any previous *p*H reguration to the filter paper impregnated with oxine and was developed with 2% acetic acid (*p*H 2.60) saturated with *n*-butanol as reported before[1], the copper zone was appeared at the inside of the iron zone because the cupric oxinate was supposed to be more stable than the ferric oxinate[5]. However, the *p*H value of the developing solvent was near the *p*H range of the complete precipitation of the ferric oxinate (*p*H 2.8～11.2[6]) and rather far from that of cupric oxinate (*p*H 5.3～14.6), so that some amount of ferric oxinate often coprecipitated in the copper zone. On the other hand, in this study, the test solution was buffered near the *p*H range of the complete precipitation of the cupric oxinate so as to prevent the coprecipitation of the ferric oxinate in the copper zone: the *p*H 5.2 buffer solution of hydrochloric acid-sodium acetate which was used in this study was high enough in *p*H as to prevent the coprecipitation of the ferric oxinate and not so high as to injure the smooth development. In this case hydrochloric acid-acetate buffer showed somewhat better result than acetic acid-acetate buffer solution[1].

2) When the 2% acetic acid (*p*H 2.60) was used as the developing solvent, the *p*H value of the solvent was in favour of the partial precipitation of copper and iron oxinates . Therefore, the velocity of the development influences the distribution of the precipitation. Moreover, the distribution may be altered by the redissolution of the

precipitation with the developing solvent which later penetrates into the precipitation. These phenomena described above are considered to be the main causes for the non-homogeneous distribution of the precipitation in the zones. Therefore, the development in this study was done at the higher *p*H. However, the development at a much higher *p*H is apt to cause the further increase of precipitation at the inner part of the zone. So the optimum *p*H of the developing solvent was sought experimentally which would be able to produce the homogeneous distribution of the precipitation of the zones and the wider applicability of the linear relation between the quantity of cation and the zone area. The development was tried with the buffer solutions of *p*H 5.2, 4.0, 3.6, 3.1 (hydrochloric acid-acetate buffer), 2% acetic acid (*p*H 2.60), and 1% acetic acid (*p*H 2.75) saturated with isobutanol. In these solvents the *p*H 3.1 buffer solution showed the best result (Fig. 1 and Fig. 2). The results obtained under the temperature range of about $10^\circ \sim 20^\circ$ C were almost the same.

The procedure described above may be recommended as a simplified quantitative procedure in the paper chromatography.

The author expresses hearty thanks to Mr. T. Kikuchi for his assistance in this study.

The general contents of this paper was published in Nippon Kagaku Zasshi, Vol. 81, p. 1271 (1960).

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ON REPRESENTATIONS OF JORDAN ALGEBRAS

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(Received April 30, 1961)

It is known that the axioms of Lie algebras characterize the structure of subspaces of associative algebras which are closed relative to the composition $ab-ba$, on the other hand the axioms of abstract Jordan algebras do not characterize the structure of subspaces of associative algebras which are closed relative to the composition $ab+ba$. With this situation in view, N. Jacobson gave a remarkable definition for the representation of Jordan algebras with an ordinary one (special representation) [9, also see 5][1] and showed that for the study of a Jordan algebra $\mathfrak{J}$ it is important to consider the structure of an associator Lie triple system of $\mathfrak{J}$ [cf. 7, 2]. Then, it seems that it is necessary to generalize the notion of representations of $\mathfrak{J}$ on the foundation of the existence of inner derivations which have meannings from the results of C. Chevalley and R. D. Schafer [4] and N. Jacobson [8].

The purpose of this paper is to define the representations of Jordan algebras in two manners and to construct the cohomology spaces which are associated with these representations. The one stands on the notion of derivations of Jordan algebras and the other stands on the notion of Lie triple derivations of associator Lie triple systems. Recently, B. Harris defined the cohomology space of special Jordan algebras and studied its properties [6].

1. **Preliminaries.**[2] We begin this section with a recalling of the basic definitions of Jordan algebras and their representations by N. Jacobson.

A *Jordan algebra*[3] over a field Φ is a non-associative algebra defined by the following identities:

$$ab=ba, \tag{1.1}$$

$$(a^2b)a=a^2(ba). \tag{1.2}$$

A subspace of an associative algebra A which is closed relative to the composition $ab=a\cdot b+b\cdot a$ is a Jordan algebra relative to the new composition ab, where $a\cdot b$ denotes the associative composition in A. A Jordan algebra isomorphic to one obtained from a subspace of an associative algebra in the above manner is called to be special. Contrary to the theory of Lie algebras, it is known that there exist non-special (exceptional) Jordan algebras.

If the characteristic of the base field Φ is different from 2, then the axioms (1.1) and (1.2) imply

1) Numbers in brackets refer to the references at the end of the paper.

2) The facts in this section will be found in the papers by A. A. Albert [1] and N. Jacobson [8, 9].

3) Except when the contrary is explicitly stated, throughout this paper we shall assume that the characteristic of the base field Φ is 0 and a Jordan algebra has a finite dimension.

$$a((bc)d)+b((ca)d)+c((ab)d)=(ab)(cd)+(bc)(ad)+(ca)(bd). \tag{1.3}$$

Conversely, if the characteristic of Φ is not 3, then we have (1.2) from (1.1) and (1.3). Therefore, a Jordan algebra may be defined by (1.1) and (1.3) in the case that the characteristic of Φ is not 2 or 3. This leads us to the following two definitions of the representations of Jordan algebras given by N. Jacobson [9].

(I) A linear mapping ρ: $a\to\rho(a)$ of a Jordan algebra $\mathfrak{J}$ into the associative algebra $E(V)$ of linear transformations of a vector space V is called a *special representation* if

$$\rho(ab)=\rho(a)\rho(b)+\rho(b)\rho(a), \tag{1.4}$$

where $(\rho(a)\rho(b))(x)=\rho(a)(\rho(b)x)$.

(II) A linear mapping ρ: $a\to\rho(a)$ of $\mathfrak{J}$ into $E(V)$ is called a *representation* if

$$[\rho(a), \rho(bc)]+[\rho(b), \rho(ca)]+[\rho(c), \rho(ab)]=0, \tag{1.5}$$

$$\begin{aligned}\rho(a)\rho(b)\rho(c)+\rho(c)\rho(b)\rho(a)+\rho(b(ca))\\ =\rho(ab)\rho(c)+\rho(bc)\rho(a)+\rho(ca)\rho(b),\end{aligned} \tag{1.6}$$

where $[\rho(a), \rho(b)]$ denotes $\rho(a)\rho(b)-\rho(b)\rho(a)$.

It is easy to see that the special representation is a representation in the sense defined in (II). In a Jordan algebra $\mathfrak{J}$, the left multiplication $L(a)$: $x\to ax(=xa)$ satisfies (1.5) and (1.6), hence L is a representation (II) in $\mathfrak{J}$. This representation is called a regular representation.

Combining (1.5) and (1.6) we have

$$\begin{aligned}\rho(a)\rho(b)\rho(c)+\rho(c)\rho(b)\rho(a)+\rho(b(ca))\\ =\rho(a)\rho(bc)+\rho(b)\rho(ca)+\rho(c)\rho(ab).\end{aligned} \tag{1.7}$$

Therefore, by using the regular representation for $\mathfrak{J}$, we see that it holds the following identity in a Jordan algebra $\mathfrak{J}$:

$$a(b(cd))+c(b(ad))+d(b(ca))=a(d(bc))+b(d(ca))+c(d(ab)). \tag{1.8}$$

Hence, if we put

$$[abc]=a(bc)-b(ac), \tag{1.9}$$

it follows that the linear mapping $\sum_i D_{(a_i,b_i)}$: $x\to\sum_i[a_ib_ix]$ is an inner derivation of a Jordan algebra, that is

$$\sum_i D_{(a_i,b_i)}(xy)=(\sum_i D_{(a_i,b_i)}x)y+x(\sum_i D_{(a_i,b_i)}y). \tag{1.10}$$

Thus, we define a more general representation than (II) as follows:

DEFINITION 1.1. If a linear mapping ρ of a Jordan algebra $\mathfrak{J}$ into the associative algebra of linear transformations of a vector space V satisfies the condition (1.7), then ρ is called a *representation (III)*.

For the representation (III), from (1.7) it follows

$$\rho([abc])=[[\rho(a),\rho(b)]\rho(c)], \tag{1.11}$$

therefore, $\rho(a)$'s are a Lie triple system and $\rho(a)+\sum_i[\rho(b_i),\rho(c_i)]$'s and $\sum_i[\rho(a_i),\rho(b_i)]$'s are Lie algebras, because

$$[[\rho(a),\rho(b)],[\rho(c),\rho(d)]]=[\rho([abc]),\rho(d)]+[\rho(c),\rho([abd])].$$

By using the relation (1.10), we obtain a simple direct proof of the result of N. Jacobson concerning a Lie triple structure of a Jordan algebra.

LEMMA 1.1. (Jacobson) *In a Jordan algebra $\mathfrak{J}$ (over a field of characteristic $\neq 2$) it holds the following relations:*

$$[aab]=0, \tag{1.12}$$

$$[abc]+[bca]+[cab]=0, \tag{1.13}$$

$$[[abc]de]+[[bad]ce]+[ba[cde]]+[cd[abe]]=0, \tag{1.14}$$

that is, $\mathfrak{J}$ is a Lie triple system with respect to the composition $[abc]=a(bc)-b(ac)$.

PROOF. (of (1.14))

$$\begin{aligned}[ab[cde]]&=[abc](de)-[abd](ce)\\&\qquad+c([abd]e+d[abe])-d([abc]e+c[abe])\\&=[[abc]de]+[c[abd]e]+[cd[abe]].\end{aligned}$$

2. **Cohomology space of Jordan algebras** (1).[4] In this section we define a cohomology space of Jordan algebras for the representation (III). Hence we base our argument on the identities (1.1) and (1.8).

Let ρ be a representation (III) of a Jordan algebra $\mathfrak{J}$ into a vector space V, and let f be a $2n$-linear mapping of $\underbrace{\mathfrak{J}\times\cdots\times\mathfrak{J}}_{2n\ times}$ into V satisfying

$$f(x_1,x_2,\cdots,x_{2n-2},x,y)-f(x_1,x_2,\cdots,x_{2n-2},y,x)=0.$$

We denote by $C^{2n}(\mathfrak{J},V)\,(n=0,1,2,\cdots)$ the vector space spanned by such $2n$-linear mappings, where $C^0(\mathfrak{J},V)=V$ by definition. Also, we consider the vector space $C^1(\mathfrak{J},V)$ spanned by linear mappings of $\mathfrak{J}$ into V.

Next, we define a linear mapping δ of $C^n(\mathfrak{J},V)$ into $C^{n+1}(\mathfrak{J},V)$ $(n=0,1)$ and of $C^{2n}(\mathfrak{J},V)$ into $C^{2n+2}(\mathfrak{J},V)$ $(n=1,2,3,\cdots)$ as follows:

$$(\delta f)(x)=\rho(x)f \qquad \text{for}\quad f\in C^0(\mathfrak{J},V), \tag{2.1}$$

$$(\delta f)(x_1,x_2)=\rho(x_1)f(x_2)+\rho(x_2)f(x_1)-f(x_1x_2) \qquad \text{for}\quad f\in C^1(\mathfrak{J},V), \tag{2.2}$$

4) cf. [3, 10].

$$
\begin{aligned}
&(\delta f)(x_1, x_2, \cdots, x_{2n+2})\\
&=(-1)^n\big[\rho(x_{2n+1})\rho(x_{2n-1})f(x_1,\cdots,x_{2n-2},x_{2n},x_{2n+2})-\rho(x_{2n+1})\rho(x_{2n})f(x_1,\cdots,x_{2n-1},x_{2n+2})\\
&\qquad+\rho(x_{2n+2})\rho(x_{2n-1})f(x_1,\cdots,x_{2n-2},x_{2n},x_{2n+1})-\rho(x_{2n+2})\rho(x_{2n})f(x_1,\cdots,x_{2n-1},x_{2n+1})\\
&\qquad-\rho(x_{2n-1})f(x_1,\cdots,x_{2n-2},x_{2n},x_{2n+1}x_{2n+2})+\rho(x_{2n})f(x_1,\cdots,x_{2n-1},x_{2n+1}x_{2n+2})\\
&\qquad+\rho(x_{2n+1})f(x_1,\cdots,x_{2n-1},x_{2n}x_{2n+2})-\rho(x_{2n+1})f(x_1,\cdots,x_{2n-2},x_{2n},x_{2n-1}x_{2n+2})\\
&\qquad+\rho(x_{2n+2})f(x_1,\cdots,x_{2n-1},x_{2n}x_{2n+1})-\rho(x_{2n+2})f(x_1,\cdots,x_{2n-2},x_{2n},x_{2n-1}x_{2n+1})\\
&\qquad-f(x_1,\cdots,x_{2n-1},x_{2n}(x_{2n+1}x_{2n+2}))+f(x_1,\cdots,x_{2n-2},x_{2n},x_{2n-1}(x_{2n+1}x_{2n+2}))\big]\\
&\quad+\sum_{k=1}^{n}(-1)^{k+1}[\rho(x_{2k-1}),\rho(x_{2k})]f(x_1,x_2,\cdots,\hat{x}_{2k-1},\hat{x}_{2k},\cdots,x_{2n+2})\\
&\quad+\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+2}(-1)^k f(x_1,x_2,\cdots,\hat{x}_{2k-1},\hat{x}_{2k},\cdots,[x_{2k-1}x_{2k}x_j],\cdots,x_{2n+2})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{for}\quad f\in C^{2n}(\mathfrak{J},V),\ n=1,2,3,\cdots,
\end{aligned}
\tag{2.3}
$$

where the sign ^ over a letter indicates that this letter is to be omitted.

For instance, if $f\in C^2(\mathfrak{J},V)$, then

$$
\begin{aligned}
&-(\delta f)(x_1, x_2, x_3, x_4)\\
&=\rho(x_3)\rho(x_1)f(x_2,x_4)-\rho(x_3)\rho(x_2)f(x_1,x_4)+\rho(x_4)\rho(x_1)f(x_2,x_3)-\rho(x_4)\rho(x_2)f(x_1,x_3)\\
&\quad-\rho(x_1)f(x_2,x_3x_4)+\rho(x_2)f(x_1,x_3x_4)+\rho(x_3)f(x_1,x_2x_4)-\rho(x_3)f(x_2,x_1x_4)\\
&\quad+\rho(x_4)f(x_1,x_2x_3)-\rho(x_4)f(x_2,x_1x_3)-[\rho(x_1),\rho(x_2)]f(x_3,x_4)\\
&\quad-f(x_1,x_2(x_3x_4))+f(x_2,x_1(x_3x_4))+f([x_1x_2x_3],x_4)+f(x_3,[x_1x_2x_4]).
\end{aligned}
$$

If $f\in C^0(\mathfrak{J},V)$, then for every $x_1, x_2\in\mathfrak{J}$

$$(\delta\delta f)(x_1,x_2)=(\rho(x_1)\rho(x_2)+\rho(x_2)\rho(x_1)-\rho(x_1x_2))f,$$

hence $\delta\delta f=0$ for all $f\in C^0(\mathfrak{J},V)$ if and only if the representation ρ reduces to the special representation.

But, we shall prove that $\delta\delta f=0$ for any $f\in C^{2n}(\mathfrak{J},V)(n=1,2,3,\cdots)$ in the sequel.

If $f\in C^2(\mathfrak{J},V)$, then this fact follows by a direct computation. In order to prove the general case, we consider the following two operations.

For $a, b\in\mathfrak{J}$, we define a linear mapping $\kappa(a,b)$ of $C^{2n}(\mathfrak{J},V)$ into $C^{2n}(\mathfrak{J},V)$ and a linear mapping $\iota(a,b)$ of $C^{2n}(\mathfrak{J},V)$ into $C^{2n-2}(\mathfrak{J},V)$ by the following formulas respectively

$$(\kappa(a,b)f)(x_1,\cdots,x_{2n})=[\rho(a),\rho(b)]f(x_1,\cdots,x_{2n})-\sum_{j=1}^{2n}f(x_1,\cdots,[abx_j],\cdots,x_{2n}),\tag{2.4}$$

$$(\iota(a,b)f)(x_1,\cdots,x_{2n-2})=f(a,b,x_1,\cdots,x_{2n-2}),\qquad n=2,3,\cdots.\tag{2.5}$$

By a direct calculation we have the following two formulas:

$$\iota(a,b)\delta f+\delta\iota(a,b)f=\kappa(a,b)f\qquad\text{for } f\in C^{2n}(\mathfrak{J},V),\quad n=2,3,\cdots,\tag{2.6}$$

$$[\kappa(a,b),\iota(c,d)]f=\iota([abc],d)f+\iota(c,[abd])f\qquad\text{for } f\in C^{2n}(\mathfrak{J},V),\quad n=2,3,\cdots.\tag{2.7}$$

Next, it holds the following relation:

(2.8) $$[\kappa(a,b),\kappa(c,d)]f=\kappa([abc],d)f+\kappa(c,[abd])f,$$
$$\text{for } f\in C^{2n}(\mathfrak{J},V),\quad n=2,3,\cdots.$$

Since for $f\in C^4(\mathfrak{J},V)$ we can prove (2.8) directly, we assume that (2.8) holds for all $f\in C^{2n}(\mathfrak{J},V)$ and let $f\in C^{2n+2}(\mathfrak{J},V)$, $n\geqq 2$, then for arbitrary $k,l\in\mathfrak{J}$ we have

$$\begin{aligned}
&\iota(k,l)([\kappa(a,b),\kappa(c,d)]-\kappa([abc],d)-\kappa(c,[abd]))f\\
&=([\kappa(a,b),\kappa(c,d)]-\kappa([abc],d)-\kappa(c,[abd]))\iota(k,l)f\\
&\quad+(\iota([cd[abk]],l)-\iota([ab[cdk]],l)+\iota([[abc]dk],l)+\iota([c[abd]k],l))f\\
&\quad+(\iota(k,[[abc]dl])+\iota(k,[c[abd]l])+\iota(k,[cd[abl]])-\iota(k,[ab[cdl]]))f\\
&=0,
\end{aligned}$$

by (1.14) and (2.7). Therefore, (2.8) holds for all $f\in C^{2n+2}(\mathfrak{J},V)$.

Moreover, it holds that

(2.9) $$\kappa(a,b)\delta f=\delta\kappa(a,b)f,\qquad \text{for } f\in C^{2n}(\mathfrak{J},V),\quad n=2,3,\cdots.$$

If $f\in C^4(\mathfrak{J},V)$, then we obtain (2.9) directly, hence we assume that (2.9) holds for all $f\in C^{2n}(\mathfrak{J},V)$. Then for $f\in C^{2n+2}(\mathfrak{J},V)$, $n\geqq 2$, and every $k,l\in\mathfrak{J}$

$$\begin{aligned}
&\iota(k,l)(\kappa(a,b)\delta-\delta\kappa(a,b))f\\
&=\kappa(a,b)\kappa(c,d)f-\kappa(c,d)\kappa(a,b)f-\kappa(a,b)\delta\iota(c,d)f+\delta\kappa(a,b)\iota(c,d)f\\
&\qquad-\iota([abc],d)\delta f-\delta\iota([abc],d)f-\iota(c,[abd])\delta f-\delta\iota(c,[abd])f\\
&=0,
\end{aligned}$$

by (2.6). Therefore, (2.9) holds for all $f\in C^{2n+2}(\mathfrak{J},V)$.

Next we see that

(2.10) $$\delta\delta f=0$$

for all $f\in C^{2n}(\mathfrak{J},V)$, $n=1,2,3,\cdots$.

We assume that (2.10) has been proved for all $f\in C^{2n}(\mathfrak{J},V)$, then for every $a,b\in\mathfrak{J}$ and $f\in C^{2n+2}(\mathfrak{J},V)$, $n\geqq 1$, by using (2.6) and (2.9) we have

$$\begin{aligned}
\iota(a,b)(\delta\delta f)&=\kappa(a,b)\delta f-\delta\iota(a,b)\delta f\\
&=\delta\delta\iota(a,b)f\\
&=0.
\end{aligned}$$

Thus we obtain the following

THEOREM 2.1. *For the operator δ defined above, it holds that $\delta\delta f=0$ for all $f\in C^{2n}(\mathfrak{J},V)$, $n=1,2,3,\cdots$. This relation holds for all $f\in C^{n}(\mathfrak{J},V)$ if and only if the representation ρ is a special representation.*

Let $Z^{2n}(\mathfrak{J}, V)$ be a subspace spanned by elements f of $C^{2n}(\mathfrak{J}, V)$ such that $\delta f=0$, and let $B^{2n}(\mathfrak{J}, V)$ be a subspace spanned by elements of $C^{2n}(\mathfrak{J}, V)$ of the form δf, then by Theorem 2.1 $B^{2n}(\mathfrak{J}, V)$ is a subspace of $Z^{2n}(\mathfrak{J}, V)$. Therefore, we can define a cohomology space $H^{2n}(\mathfrak{J}, V)$ of order $2n$ of a Jordan algebra $\mathfrak{J}$ as the factor space $Z^{2n}(\mathfrak{J}, V)/B^{2n}(\mathfrak{J}, V)$, where $n=1, 2, 3, \cdots$.

REMARK 2.1. For $f\in C^1(\mathfrak{J}, V)$ we have $\delta\delta f=0$, because

$$
\begin{aligned}
&(\delta\delta f)(x_1, x_2, x_3, x_4)\\
&=(\rho(x_3)\rho(x_2)\rho(x_4)+\rho(x_4)\rho(x_2)\rho(x_3)+\rho(x_2(x_3x_4))\\
&\qquad\qquad -\rho(x_2)\rho(x_3x_4)-\rho(x_3)\rho(x_4x_2)-\rho(x_4)\rho(x_2x_3))f(x_1)\\
&\quad -(\rho(x_3)\rho(x_1)\rho(x_4)+\rho(x_4)\rho(x_1)\rho(x_3)+\rho(x_1(x_3x_4))\\
&\qquad\qquad -\rho(x_1)\rho(x_3x_4)-\rho(x_3)\rho(x_4x_1)-\rho(x_4)\rho(x_1x_3))f(x_2)\\
&\quad +([[\rho(x_1), \rho(x_2)]\rho(x_4)]-\rho([x_1x_2x_4]))f(x_3)\\
&\quad +([[\rho(x_1), \rho(x_2)]\rho(x_3)]-\rho([x_1x_2x_3]))f(x_4)\\
&\quad -f(x_1(x_2(x_3x_4)))+f(x_2(x_1(x_3x_4)))+f([x_1x_2x_3]x_4)+f(x_3[x_1x_2x_4])\\
&=0.
\end{aligned}
$$

REMARK 2. 2. For $f\in C^0(\mathfrak{J}, V)$ we can define the coboundary operator δ of $C^0(\mathfrak{J}, V)$ into $C^2(\mathfrak{J}, V)$ as follows:

$$(\delta f)(x_1, x_2)=(\rho(x_1)\rho(x_2)+\rho(x_2)\rho(x_1)-\rho(x_1x_2))f.$$

Then, $\delta\delta f=0$, hence in this case we can define the cohomology space $H^{2n}(\mathfrak{J}, V)$ for all non-negative integer n, and the gap in Theorem 2.1 will be filled.

3. **Cohomology space of Jordan algebras** (2). In this section we define a cohomology space of Jordan algebras as associator Lie triple systems. For this purpose we generalize the representation (II) of Jordan algebras.

DEFINITION 3.1. Let $\rho: a\to\rho(a)$ be a linear mapping of a Jordan algebra $\mathfrak{J}$ into the associative algebra of linear transformations of a vector space V. This mapping is called a *representation* (*IV*) of $\mathfrak{J}$ if $\rho(a)$ satisfies the following relations:

$$
\begin{aligned}
(3.1)\quad &(\rho(cd)-\rho(c)\rho(d))(\rho(ab)-\rho(a)\rho(b))-(\rho(bd)-\rho(b)\rho(d))(\rho(ac)-\rho(a)\rho(c))\\
&\qquad =\rho(a[bcd])-\rho(a)\rho([bcd])-[\rho(b), \rho(c)](\rho(ad)-\rho(a)\rho(d)),\\
(3.2)\quad &[[\rho(a), \rho(b)]\ \rho(cd)-\rho(c)\rho(d)]\\
&\qquad =\rho([a, b, cd])-\rho([abc])\rho(d)-\rho(c)\rho([abd]).
\end{aligned}
$$

Then we have the following

THEOREM 3.1. *Let $\rho: a\to\rho(a)$ be a representation* (II) *of a Jordan algebra $\mathfrak{J}$ into a vector space, then $\rho(a)$ satisfies the conditions* (3.1) *and* (3.2).

PROOF.

$$
\begin{aligned}
&(\rho(cd)-\rho(c)\rho(d))(\rho(ab)-\rho(a)\rho(b))-(\rho(bd)-\rho(b)\rho(d))(\rho(ac)-\rho(a)\rho(c))\\
&\quad -\rho(a[bcd])+\rho(a)\rho([bcd])+[\rho(b),\rho(c)](\rho(ad)-\rho(a)\rho(d))\\
&=\rho(cd)(\rho(ab)-\rho(a)\rho(b))-\rho(bd)(\rho(ac)-\rho(a)\rho(c))-\rho(a[bcd])+\rho(a)\rho([bcd])\\
&\qquad -\rho(c)(\rho(a(bd))-\rho(a)\rho(bd))+\rho(b)(\rho(a(cd))-\rho(a)\rho(cd))\\
&=\rho(ab)\rho(cd)+\rho(b(cd))\rho(a)+\rho(a(cd))\rho(b)-\rho(ac)\rho(bd)\\
&\quad -\rho(c(bd))\rho(a)-\rho(a(bd))\rho(c)-\rho(cd)\rho(a)\rho(b)+\rho(bd)\rho(a)\rho(c)\\
&\qquad -\rho(a[bcd])+\rho(c)\rho(a)\rho(bd)-\rho(b)\rho(a)\rho(cd)\\
&=0,
\end{aligned}
$$

hence, (3.1) was proved. Next, we shall prove (3.2).

$$
\begin{aligned}
&[\rho(cd)-\rho(c)\rho(d),\ [\rho(a),\rho(b)]]+\rho([a,b,cd])-\rho([abc])\rho(d)-\rho(c)(\rho[abd])\\
&=[\rho(cd)[\rho(a),\rho(b)]]+\rho([a,b,cd])\\
&\qquad +\rho(c)([[\rho(a),\rho(b)]\rho(d)]-\rho([abd]))\\
&\qquad\quad +([[\rho(a),\rho(b)]\rho(c)]-\rho([abc]))\rho(d)\\
&=0.
\end{aligned}
$$

Now, for the representation (IV) $\rho\colon a\rightarrow\rho(a)$ of a Jordan algebra $\mathfrak{J}$ into a vector space V, if we put

$$
\theta(a,b)=\rho(ab)-\rho(a)\rho(b),
$$

$$
D(a,b)=\theta(b,a)-\theta(a,b),
$$

then the conditions (3.1) and (3.2) can be rewritten as

(3.1)′ $$\theta(c,d)\theta(a,b)-\theta(b,d)\theta(a,c)-\theta(a,[bcd])+D(b,c)\theta(a,d)=0,$$

(3.2)′ $$[D(a,b),\theta(c,d)]=\theta([abc],d)+\theta(c,[abd]).$$

(3.1)′ and (3.2)′ are the conditions for the representation of Lie triple systems in [10], hence by Lemma 1.1 the representation space V becomes a $\mathfrak{T}$-module [10, Definition 2]. Especially, for a regular representation in a Jordan algebra $\mathfrak{J}$, $D(a,b)$ is an inner derivation in $\mathfrak{J}$.

Let $C^n(\mathfrak{J},V)$ be a vector space spanned by n-linear mappings f of $\underbrace{\mathfrak{J}\times\cdots\times\mathfrak{J}}_{n\ times}$ into a $\mathfrak{T}$-module V such that

$$
f(x_1,x_2,\cdots,x_{n-3},x,x,x_n)=0
$$

and

$$
f(x_1,x_2,\cdots,x_{n-3},x,y,z)+f(x_1,x_2,\cdots,x_{n-3},y,z,x)+f(x_1,x_2,\cdots,x_{n-3},z,x,y)=0,
$$

where we define $C^0(\mathfrak{J},V)=V$.

A linear mapping δ of $C^n(\mathfrak{J},V)$ into $C^{n+2}(\mathfrak{J},V)$ is defined by the following formulas:

$$(\delta f)(x_1, x_2) = \theta(x_1, x_2) f \qquad \text{for} \quad f \in C^0(\mathfrak{J}, V), \tag{3.3}$$

$$\begin{aligned}
&(\delta f)(x_1, x_2, \cdots, x_{2n+1}) \\
&= \theta(x_{2n}, x_{2n+1}) f(x_1, x_2, \cdots, x_{2n-1}) - \theta(x_{2n-1}, x_{2n+1}) f(x_1, x_2, \cdots, x_{2n-2}, x_{2n}) \\
&\quad + \sum_{k=1}^{n} (-1)^{n+k} D(x_{2k-1}, x_{2k}) f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1}) \\
&\quad + \sum_{k=1}^{n} \sum_{j=2k+1}^{2n+1} (-1)^{n+k+1} f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1} x_{2k} x_j], \cdots, x_{2n+1}) \\
&\qquad\qquad \text{for } f \in C^{2n-1}(\mathfrak{J}, V), \quad n = 1, 2, 3, \cdots,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&(\delta f)(y, x_1, x_2, \cdots, x_{2n+1}) \\
&= \theta(x_{2n}, x_{2n+1}) f(y, x_1, x_2, \cdots, x_{2n-1}) - \theta(x_{2n-1}, x_{2n+1}) f(y, x_1, x_2, \cdots, x_{2n-2}, x_{2n}) \\
&\quad + \sum_{k=1}^{n} (-1)^{n+k} D(x_{2k-1}, x_{2k}) f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1}) \\
&\quad + \sum_{k=1}^{n} \sum_{j=2k+1}^{2n+1} (-1)^{n+k+1} f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1} x_{2k} x_j], \cdots, x_{2n+1}) \\
&\qquad\qquad \text{for } f \in C^{2n}(\mathfrak{J}, V), \quad n = 1, 2, 3, \cdots,
\end{aligned} \tag{3.5}$$

where the sign ^ over a letter indicates that this letter is to be omitted. Then, from [10, Theorem 1] we have $\delta\delta f = 0$ for any $f \in C^n(\mathfrak{J}, V)(n = 0, 1, 2, \cdots)$, hence we can define a cohomology space $H^n(\mathfrak{J}, V)$ of order n for a Jordan algebra $\mathfrak{J}$ as the factor space $Z^n(\mathfrak{J}, V)/B^n(\mathfrak{J}, V)$, where $Z^n(\mathfrak{J}, V)$ is a subspace of $C^n(\mathfrak{J}, V)$ spanned by f such that $\delta f = 0$ and $B^n(\mathfrak{J}, V) = \delta C^{n-2}(\mathfrak{J}, V)$.

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SOME STUDIES ON THE ORTHOGONALITY RELATIONS FOR GROUP CHARACTERS

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(Received April 24, 1961)

R. BRAUER, in his paper [1], gave an important refinement of some of the orthogonality relations for group characters.[1] He and M. OSIMA gave independently another refinement, [9].[2] In [10], [5] ([4]) and [6], other refinements were discussed. In the present note, we shall extend the results to a more general case.

The author wishes to thank Professor M. OSIMA and Professor T. NAKAYAMA for their helpful suggestions.

1. Preliminaries.

Let $\mathfrak{G}$ be a group of finite order g and let $p_1, p_2, \cdots, p_r$ be the rational primes dividing g: $g=p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}, a_i>0$. We shall denote by $K_1, K_2, \cdots, K_n$ the classes of conjugate elements in $\mathfrak{G}$; $\mathfrak{G}$ has n distinct absolutely irreducible (ordinary) characters, $\chi_1, \chi_2, \cdots, \chi_n$.

1. 1. We shall consider the subsets of the set $\{p_1, p_2, \cdots, p_r\}$: Π_0=empty set, $\Pi_1=\{p_1, p_2, \cdots, p_r\}$, $\Pi_2, \cdots, \Pi_{2^r-1}$. If $\bigcup$ and $\bigcap$ respectively mean the set theoretical union and intersection, then the subsets Π_f form a lattice with respect to two operations $\bigcup$ and $\bigcap$, which has the maximum element Π_1 and the minimum element Π_0. We put $\pi_f=\prod_{p_i \in \Pi_f} p_i^{a_i}$ $(1 \leqq f \leqq 2^r-1)$ and $\pi_0=1$.

For each Π_f, we can find the maximum one, say $\mathfrak{N}_{\Pi_f}$, among the normal subgroups of $\mathfrak{G}$ whose orders are prime to π_f. The following facts are easily seen:

1) $\mathfrak{N}_{\Pi_0}=\mathfrak{G}$, $\mathfrak{N}_{\Pi_1}=\{1\}$.
2) $\Pi_f \supseteqq \Pi_h$ implies $\mathfrak{N}_{\Pi_f} \subseteqq \mathfrak{N}_{\Pi_h}$.
3) $\mathfrak{N}_{\Pi_f \cup \Pi_h}=\mathfrak{N}_{\Pi_f} \cap \mathfrak{N}_{\Pi_h}$, $\mathfrak{N}_{\Pi_f \cap \Pi_h} \supseteqq \mathfrak{N}_{\Pi_f} \cdot \mathfrak{N}_{\Pi_h}$.

We shall call an element P of $\mathfrak{G}$ a Π_f-element if and only if its order is not divisible by any rational prime not belonging to Π_f, while we shall call an element R of $\mathfrak{G}$ a Π_f-regular element if and only if its order is prime to π_f. As is well known, an element G of $\mathfrak{G}$ can be uniquely expressed as a product of two commutative elements P and R where P is a *Π_f-element*, while R is a *Π_f-regular* element; we shall call P the Π_f-factor of G and R the Π_f-regular factor of G. If Π_f consists of one rational prime p, then "Π_f-" will become "p-".

The *Π_f-section* of a Π_f-element P in $\mathfrak{G}$ is the set of all elements of $\mathfrak{G}$ whose Π_f-factors are conjugate to P in $\mathfrak{G}$; we shall denote by $\mathfrak{S}^{\Pi_f}(G)$ the Π_f-section of $\mathfrak{G}$ represented by an element G. As is easily seen, each Π_f-section of $\mathfrak{G}$ is a collection

1) Cf. also [2].
2) Cf. also [3].

of classes K_ν of $\mathfrak{G}$. We choose a complete system of representatives, $P_1^{\Pi_f}=1, P_2^{\Pi_f}, \cdots, P_{s_f}^{\Pi_f}$, for the classes K_ν of $\mathfrak{G}$ which consist of Π_f-elements. It is easily seen that the elements of $\mathfrak{G}$ are distributed into the s_f Π_f-sections $\mathfrak{S}_\nu^{\Pi_f}=\mathfrak{S}^{\Pi_f}(P_\nu^{\Pi_f})$ such that each element of $\mathfrak{G}$ belongs to exactly one $\mathfrak{S}_\nu^{\Pi_f}$. If Π_f consists of one rational prime p, then the Π_f-sections of $\mathfrak{G}$ are the p-sections of $\mathfrak{G}$. On the other hand, if Π_f consists of $r-1$ rational primes, then the Π_f-sections of $\mathfrak{G}$ are the p-regular sections of $\mathfrak{G}$, where p is the rational prime in Π_1 not belonging to Π_f.

The following facts are easily seen:

4) Only Π_0-section of $\mathfrak{G}$ is $\mathfrak{G}$ itself, while the Π_1-sections of $\mathfrak{G}$ are the classes K_ν of $\mathfrak{G}$.

5) If $\Pi_f \supseteqq \Pi_h$, then each Π_h-section of $\mathfrak{G}$ is a collection of Π_f-sections of $\mathfrak{G}$.

1. 2. The *Π_f-block* $\boldsymbol{B}^{\Pi_f}(\chi_i)$ of irreducible characters of $\mathfrak{G}$ which is represented by an irreducible character χ_i is the set of all irreducible characters χ_j of $\mathfrak{G}$ such that each χ_j is connected to χ_i by a chain of irreducible characters of $\mathfrak{G}$,

$$\chi_i, \chi_\lambda, \cdots, \chi_\rho, \chi_j,$$

in which any two consecutive χ_α and χ_β belong to a p-block of $\mathfrak{G}$, where $p=p_{\varphi(\alpha,\beta)}$ is a rational prime in Π_f. It is understood that each irreducible character χ_i of $\mathfrak{G}$ itself forms a Π_0-block of $\mathfrak{G}$. We denote by $\boldsymbol{B}_1^{\Pi_f}, \boldsymbol{B}_2^{\Pi_f}, \cdots, \boldsymbol{B}_{t_f}^{\Pi_f}$ the Π_f-blocks of $\mathfrak{G}$. If Π_f consists of one rational prime p, then the Π_f-blocks of $\mathfrak{G}$ are the p-blocks of $\mathfrak{G}$. On the other hand, if Π_f consists of $r-1$ rational primes and p is the rational prime in Π_1 not belonging to Π_f,then the Π_f-blocks of $\mathfrak{G}$ are the p-complementary blocks of $\mathfrak{G}$.

The following facts are easily seen:

1) If $\Pi_f \supseteqq \Pi_h$, then each Π_f-block of $\mathfrak{G}$ is a collection of Π_h-blocks of $\mathfrak{G}$.

2) $\boldsymbol{B}^{\Pi_f \cup \Pi_h}(\chi_i) \supseteqq \boldsymbol{B}^{\Pi_f}(\chi_i) \cup \boldsymbol{B}^{\Pi_h}(\chi_i)$.

1. 3. Let $\mathfrak{N}$ be a normal subgroup of $\mathfrak{G}$ and let $\mathfrak{B}_1^{\mathfrak{N}}, \mathfrak{B}_2^{\mathfrak{N}}, \cdots, \mathfrak{B}_{n(\mathfrak{N})}^{\mathfrak{N}}$ be the $\mathfrak{N}$-blocks[3] of irreducible characters χ_i of $\mathfrak{G}$. It is well known that the classes of associated irreducible characters θ_λ of $\mathfrak{N}$ in $\mathfrak{G}$ one-one correspond to the $\mathfrak{N}$-blocks of $\mathfrak{G}$; we shall denote by $\mathfrak{U}_\sigma^{\mathfrak{N}}$ the class of associated irreducible characters of $\mathfrak{N}$ which corresponds to $\mathfrak{B}_\sigma^{\mathfrak{N}}$. If we denote by Φ_σ the sum of all irreducible characters θ_λ in $\mathfrak{U}_\sigma^{\mathfrak{N}}$, then for each irreducible character χ_i in $\mathfrak{B}_\sigma^{\mathfrak{N}}$

$$\chi_i(N)=s_{i\sigma}\Phi_\sigma(N) \qquad (N \in \mathfrak{N}),$$

where $s_{i\sigma}$ is a positive rational integer.

The following facts are well known:

1) Two irreducible characters χ_i and χ_j of $\mathfrak{G}$ belong to a same $\mathfrak{N}$-block of $\mathfrak{G}$ if and only if $\chi_i(N)/\chi_i(1)=\chi_j(N)/\chi_j(1)$ holds for all elements N of $\mathfrak{N}$.

2) If we denote by $\varphi^{\mathfrak{G}}$ the character of $\mathfrak{G}$ which is induced by a character φ of $\mathfrak{N}$, then

$$\theta_\lambda^{\mathfrak{G}}(G)=\sum_{\chi_i \in \mathfrak{B}_\sigma^{\mathfrak{N}}} s_{i\sigma}\chi_i(G) \qquad (G \in \mathfrak{G}),$$

3) Cf. [7]. Cf. also [5] or [4].

where θ_λ is an irreducible character in $\mathfrak{U}_\sigma^{\mathfrak{N}}$.

3) If $\mathfrak{M} \supseteqq \mathfrak{N}$, then every $\mathfrak{N}$–block of $\mathfrak{G}$ is a collection of $\mathfrak{M}$–blocks of $\mathfrak{G}$.

4) If the order of $\mathfrak{N}$ is prime to π_f, then every $\mathfrak{N}$–block of $\mathfrak{G}$ is a collection of $\mathfrak{N}_{\Pi_f}$–blocks of $\mathfrak{G}$.

5) If $\Pi_f \supseteqq \Pi_h$, then every $\mathfrak{N}_{\Pi_f}$–block of $\mathfrak{G}$ is a collection of $\mathfrak{N}_{\Pi_h}$–blocks of $\mathfrak{G}$.

2. Π_f–blocks.

2. 1. Let Ω be the field of g–th roots of unity and let Z be the center of the group ring Γ of $\mathfrak{G}$ over Ω. We denote by e_i the primitive idempotent of Z which is associated with an irreducible character χ_i of $\mathfrak{G}$:

$$e_i = \frac{1}{g} \sum_{\nu=1}^{n} \chi_i(1)\chi_i(G_\nu^{-1})K_\nu,$$

where G_ν is a representative element of K_ν and each class K_ν also denotes the sum of all its elements.

Let p be a rational prime in Π_1 and let $\mathfrak{o}_\mathfrak{p}$ be the ring of all $\mathfrak{p}$–integers in Ω, where $\mathfrak{p}$ is a prime ideal divisor of p in Ω. It is well known that if, for each p–block B_τ of $\mathfrak{G}$, we denote by E_τ the idempotent of Z which is associated with B_τ, then the idempotents E_τ are the primitive idempotents of the center $Z_\mathfrak{p}$ of the group ring $\Gamma_\mathfrak{p}$ of $\mathfrak{G}$ over $\mathfrak{o}_\mathfrak{p}$ and that each E_τ is a linear combination of p-regular classes K_ν of $\mathfrak{G}$.[4]

We denote by $\boldsymbol{E}_\delta^{\Pi_f}$ the idempotent of Z which is associated with a Π_f-block $\boldsymbol{B}_\delta^{\Pi_f}$ of $\mathfrak{G}$:

$$\boldsymbol{E}_\delta^{\Pi_f} = \sum_{\chi_i \in B_\delta^{\Pi_f}} e_i.$$

We set

$$\boldsymbol{E}_\delta^{\Pi_f} = \sum_\nu \beta_{\delta,\nu}^{\Pi_f} K_\nu.$$

〔2.1.A〕 *The Π_f–blocks $\boldsymbol{B}_\delta^{\Pi_f}$ of $\mathfrak{G}$ are characterized as the minimal sets $\boldsymbol{B}$ of irreducible characters χ_i of $\mathfrak{G}$ such that (a) each $\boldsymbol{B}$ is a collection of q blocks of $\mathfrak{G}$ for any rational prime q in Π_f, (b) each $\boldsymbol{B}$ is not vacuous.*

〔2.1.B〕 *$\beta_{\delta,\nu}^{\Pi_f}$ can be different from zero only for Π_f–regular classes K_ν of $\mathfrak{G}$ (i. e. classes K_ν of $\mathfrak{G}$ which consist of Π_f regular elements). The $\beta_{\delta,\nu}^{\Pi_f}$ multiplied by g/π_f are algebraic integers.*

PROOF. Since $\boldsymbol{B}_\delta^{\Pi_f}$ is a collection of q–blocks of $\mathfrak{G}$ for each rational prime q in Π_f, $\beta_{\delta,\nu}^{\Pi_f}$ can differ from zero only for Π_f-regular classes K_ν of $\mathfrak{G}$. Since, further, the $\beta_{\delta,\nu}^{\Pi_f}$ multiplied by g are algebraic integers, the $g/\pi_f \cdot \beta_{\delta,\nu}^{\Pi_f}$ are algebraic integers.

〔2.1.C〕 *If $\boldsymbol{B}$ is a set of irreducible characters χ_i of $\mathfrak{G}$ such that the coefficients β_ν of*

$$\omega = \sum_{\chi_i \in \boldsymbol{B}} e_i = \sum_\nu \beta_\nu K_\nu$$

4) Cf. [8].

are algebraic integers, then $\boldsymbol{B}$ *is a collection of* Π_f*-blocks* $\boldsymbol{B}_\delta^{\Pi_f}$ *of* $\mathfrak{G}$, *where* α *is a product of powers of the rational primes in* Π_1 *not belonging to* Π_f.

PROOF. If the β_ν are algebraic integers for an α, then $\boldsymbol{B}$ is a collection of q-blocks of $\mathfrak{G}$ for each rational prime q in Π_f. Hence, it is easily seen from [2.1.A] that $\boldsymbol{B}$ is a collection of Π_f-blocks of $\mathfrak{G}$.

As a special case, we have

[2.1.D] *Only* Π_1*-block of* $\mathfrak{G}$ *is the set of all irreducible characters* χ_i *of* $\mathfrak{G}$. *Only primitive idempotent of the center of the group ring of* $\mathfrak{G}$ *over the ring of all rational integers is the identity* 1.

2. 2. Let $\Pi_f=\{q_1, q_2, \cdots, q_u\}$ be an arbitrarily given subset of Π_1 and let P be a Π_f-element of $\mathfrak{G}$. We consider the normalizer, $\tilde{\mathfrak{G}}$, of P in $\mathfrak{G}$. If $\tilde{K}_1, \tilde{K}_2, \cdots, \tilde{K}_{\tilde{k}}$ are the Π_f-regular classes of $\tilde{\mathfrak{G}}$ (strictly speaking, the $\tilde{\Pi}_f$-regular classes of $\tilde{\mathfrak{G}}$ where $\tilde{\Pi}_f$ is the set of all rational primes in Π_f which divide the order $\tilde{g}$ of $\tilde{\mathfrak{G}}$) then, for any different $\alpha, \beta\,(1\leqq\alpha, \beta\leqq\tilde{k})$, $P\tilde{K}_\alpha$ and $P\tilde{K}_\beta$ cannot be contained in a same class K_ν of $\mathfrak{G}$. Hence, arranging the classes K_ν of $\mathfrak{G}$ in a suitable order, we may assume that each $P\tilde{K}_\alpha$ is contained in a class K_α of $\mathfrak{G}\,(\alpha=1, 2, \cdots, \tilde{k})$; $K_1, K_2, \cdots, K_{\tilde{k}}$ are the classes of $\mathfrak{G}$ which are contained in the Π_f-section $\mathfrak{S}^{\Pi_f}(P)$ of P in $\mathfrak{G}$.

It is well known that P is uniquely expressed as a product

$$P=Q_1, Q_2, \cdots, Q_u$$

where Q_i is the q_i-factor of $P\,(1\leqq i\leqq u)$. First, for a Π_f-block $\boldsymbol{B}_\delta^{\Pi_f}$ of $\mathfrak{G}$, we consider the collection $\boldsymbol{B}_{(\delta)}^{\Pi_f}(Q_1)$ of q_1-blocks $B_\rho^{Q_1}$ of the normalizer $\mathfrak{G}_1$ of Q_1 in $\mathfrak{G}$ such that each $B_\rho^{Q_1}$ is associated (in Brauer's sense) with a q_1-block of $\mathfrak{G}$ which is contained in $\boldsymbol{B}_\delta^{\Pi_f}$. It is easy to see that $\boldsymbol{B}_{(\delta)}^{\Pi_f}(Q_1)$ is a collection of Π_f-blocks of $\mathfrak{G}_1$[5]. Secondly, if we consider the collection $\boldsymbol{B}_{(\delta)}^{\Pi_f}(Q_1Q_2)$ of q_2-blocks $B_\mu^{Q_2}$ of the normalizer $\mathfrak{G}_2$ of Q_1Q_2 in $\mathfrak{G}$ such that each $B_\mu^{Q_2}$ is associated with a q_2-block of $\mathfrak{G}_1$ contained in $\boldsymbol{B}_{(\delta)}^{\Pi_f}(Q_1)$, then $\boldsymbol{B}_{(\delta)}^{\Pi_f}(Q_1Q_2)$ is a collection of Π_f-blocks of $\mathfrak{G}_2$. Continuing this process, we have finally a collection $\tilde{\boldsymbol{B}}_{(\delta)}^{\Pi_f}=\boldsymbol{B}_{(\delta)}^{\Pi_f}(P)$ of Π_f-blocks $\tilde{\boldsymbol{B}}_\gamma^{\Pi_f}$ of $\tilde{\mathfrak{G}}$.

If we denote by $\tilde{\boldsymbol{E}}_{(\delta)}^{\Pi_f}$ the idempotent of the center $\tilde{Z}$ of the group ring of $\tilde{\mathfrak{G}}$ over Ω which is associated with $\tilde{\boldsymbol{B}}_{(\delta)}^{\Pi_f}$, then we have

[2.2.A] *For* $\alpha=1, 2, \cdots, \tilde{k}$, *we may write*

$$\tilde{K}_\alpha\,\tilde{\boldsymbol{E}}_{(\delta)}^{\Pi_f}=\sum_{\beta=1}^{\tilde{k}}\beta_{\delta,\alpha\beta}^{\Pi_f}\tilde{K}_\beta$$

and

$$K_\alpha\boldsymbol{E}_\delta^{\Pi_f}=\sum_{\beta=1}^{\tilde{k}}\beta_{\delta,\alpha\beta}^{\Pi_f}K_\beta$$

with the same coefficients $\beta_{\delta,\alpha\beta}^{\Pi_f}$.

5) Cf. [6].

2. 3.[6] According to [2.2.A], we obtain the following refinements of some of the orthogonality relations for group characters.

[2.3.A] *If* L *and* M *are two elements of* $\mathfrak{G}$ *which belong to different* Π_f*-sections of* $\mathfrak{G}$, *then*

$$\sum_{\chi_i \in B} \chi_i(L)\chi_i(M^{-1}) = 0$$

for each Π_f*-block* $B = B_\delta^{\Pi_f}$ *of* $\mathfrak{G}$.

[2.3.B] *If* χ_i *and* χ_j *are two irreducible characters of* $\mathfrak{G}$ *which belong to different* Π_f*-blocks of* $\mathfrak{G}$, *then*

$$\sum_{G \in \mathfrak{S}} \chi_i(G)\chi_j(G^{-1}) = 0$$

for each Π_f*-section* $\mathfrak{S} = \mathfrak{S}_\gamma^{\Pi_f}$ *of* $\mathfrak{G}$.

Combining 5) in 1.1 with Theorem 3 in [8], we obtain

[2.3.C] *If* B *is a set of irreducible characters* χ_i *of* $\mathfrak{G}$ *such that*

$$\sum_{\chi_i \in B} \chi_i(L)\chi_j(M^{-1}) = 0$$

for any two elements L *and* M *of* $\mathfrak{G}$ *which belong to different* Π_f*-sections of* $\mathfrak{G}$, *then* B *is a collection of* Π_f*-blocks of* $\mathfrak{G}$.

REMARK. Let $\mathfrak{S}$ be a collection of classes K_ν of $\mathfrak{G}$. $\mathfrak{S}$ is not always a collection of Π_f-sections of $\mathfrak{G}$, if

$$\sum_{G \in \mathfrak{S}} \chi_i(G)\chi_j(G^{-1}) = 0$$

holds for any two irreducible characters χ_i and χ_j of $\mathfrak{G}$ which belong to different Π_f-blocks of $\mathfrak{G}$.

2. 4. Let X be the character ring of $\mathfrak{G}$ over Ω:

$$X = \Omega\chi_1 + \Omega\chi_2 + \cdots + \Omega\chi_n.$$

The identity of X is the sum of n mutually orthogonal primitive idempotents $d_1, d_2, \cdots, d_n$ of X:

$$d_\mu(G_\nu) = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases}$$

where G_ν is a representative element of K_ν $(\nu = 1, 2, \cdots, n)$. As is well known, d_μ is given by

$$d_\mu = \frac{1}{g} \sum_{i=1}^{n} c_\mu \chi_i(G_\mu^{-1})\chi_i, \tag{2}$$

where c_μ is the number of elements in K_μ.

6) Cf. Remark in [6].

It is well known that if $S_1, S_2, \cdots, S_t$ are the p-regular sections of $\mathfrak{G}$ for a rational prime p, then the idempotents δ_γ of X associated with the p-regular sections S_γ are the mutually orthogonal primitive idempotents of the character ring

$$X_0 = \mathfrak{o}_\mathfrak{p}\chi_1 + \mathfrak{o}_\mathfrak{p}\chi_2 + \cdots + \mathfrak{o}_\mathfrak{p}\chi_n$$

of $\mathfrak{G}$ over the ring $\mathfrak{o}_\mathfrak{p}$ of all $\mathfrak{p}$-integers in Ω, where $\mathfrak{p}$ is a prime ideal divisor of p in Ω.[7]

For each Π_f-section $\mathfrak{S}_\gamma^{\Pi_f}$ of $\mathfrak{G}$, we consider the idempotent $\varepsilon_\gamma^{\Pi_f}$ of X which is associated with $\mathfrak{S}_\gamma^{\Pi_f}$:

$$\varepsilon_\gamma^{\Pi_f} = \sum_{K_\nu \subseteq \mathfrak{S}_\gamma^{\Pi_f}} d_\nu.$$

[2.4.A] *The Π_f-sections $\mathfrak{S}_\gamma^{\Pi_f}$ of $\mathfrak{G}$ are characterized as the minimal collections $\mathfrak{S}$ of classes K_ν of $\mathfrak{G}$ such that (a) each $\mathfrak{S}$ is a collection of q-regular sections of $\mathfrak{G}$ for any rational prime q not belonging to Π_f, (b) each $\mathfrak{S}$ is not vacuous.*

If we set

$$\varepsilon_\gamma^{\Pi_f} = \sum_i \alpha_{\gamma,i}^{\Pi_f} \chi_i, \tag{3}$$

then we have

[2.4.B] *$\alpha_{\gamma,i}^{\Pi_f}$ can be different from zero only for characters χ_i which belong to the Π_f-block $B_1^{\Pi_f}$ containing the 1-character χ_1. The $\alpha_{\gamma,i}^{\Pi_f}$ multiplied by π_f are algebraic integers.*

[2.4.C] *If $\mathfrak{S}$ is a collection of classes K_ν of $\mathfrak{G}$ such that the coefficients α_i of $\beta\cdot\sum_{K_\nu \subseteq \mathfrak{S}} d_\nu = \sum_i \alpha_i\chi_i$ are algebraic integers, then $\mathfrak{S}$ is a collection of Π_f-sections $\mathfrak{S}_\gamma^{\Pi_f}$ of $\mathfrak{G}$, where β is a product of powers of the rational primes in Π_f.*

3. Blocks with regard to normal subgroups.

3.1. Let $\mathfrak{N}$ be a normal subgroup of $\mathfrak{G}$ whose order is prime to π_f. We consider the idempotents $\Delta_\sigma^{\mathfrak{N}}$ of Z which are associated with the $\mathfrak{N}$-blocks $\mathfrak{B}_\sigma^{\mathfrak{N}}$ of $\mathfrak{G}$:

$$\Delta_\sigma^{\mathfrak{N}} = \sum_{\chi_i \in \mathfrak{B}_\sigma^{\mathfrak{N}}} e_i.$$

We set

$$\Delta_\sigma^{\mathfrak{N}} = \sum_\nu a_{\sigma,\nu}^{\mathfrak{N}} K_\nu, \tag{4}$$

where $a_{\sigma,\nu}^{\mathfrak{N}} \in \Omega$. According to facts mentioned in 1.3, we have

[3.1.A] *$a_{\sigma,\nu}^{\mathfrak{N}}$ can be different from zero only for classes K_ν which are contained in $\mathfrak{N}$. The $(\mathfrak{N}:1)a_{\sigma,\nu}^{\mathfrak{N}}$ are algebraic integers.*

[3.1.B] *If $\mathfrak{B}$ is a set of irreducible characters χ_i of $\mathfrak{G}$ such that $\sum_{\chi_i \in \mathfrak{B}} e_i$ is a linear combination of classes K_ν contained in $\mathfrak{N}$, then $\mathfrak{B}$ is a collection of $\mathfrak{N}$-blocks $\mathfrak{B}_\sigma^{\mathfrak{N}}$ of $\mathfrak{G}$.*

7) Cf. [11], [12]. Cf. also [6].

Combining [3. 1. A] with [2. 1. C], we have

[3. 1. C] *If $\mathfrak{N}$ is a normal subgroup of $\mathfrak{G}$ whose order is prime to π_f, then each $\mathfrak{N}$-block $\mathfrak{B}^{\mathfrak{N}}_\sigma$ of $\mathfrak{G}$ is a collection of π_f-blocks $B^{\Pi_f}_\delta$ of $\mathfrak{G}$.*

We set

$$K_\mu \Delta^{\mathfrak{N}}_\sigma = \sum_\nu a^{\mathfrak{N}}_{\sigma,\mu\nu} K_\nu, \tag{5}$$

where $a^{\mathfrak{N}}_{\sigma,\mu\nu} \in \Omega$. If $\mathfrak{M}$ is a normal subgroup of $\mathfrak{G}$ which contains $\mathfrak{N}$, then $a^{\mathfrak{N}}_{\sigma,\mu\nu}$ can differ from zero only when either both K_μ and K_ν are contained in $\mathfrak{M}$ or when both are not contained in $\mathfrak{M}$. Thus we have

[3. 1. D] *If $\mathfrak{M}$ is a normal subgroup of $\mathfrak{G}$ which contains $\mathfrak{N}$ and if exactly one of two elements L and M of $\mathfrak{G}$ belongs to $\mathfrak{M}$, then*

$$\sum_{\chi_i \in \mathfrak{B}} \chi_i(L)\chi_i(M^{-1}) = 0$$

for each $\mathfrak{N}$-block $\mathfrak{B} = \mathfrak{B}^{\mathfrak{N}}_\sigma$ of $\mathfrak{G}$.

3. 2. Let P be a Π_f-element of $\mathfrak{G}$ and $\mathfrak{N}$ a normal subgroup of $\mathfrak{G}$ whose order is prime to π_f. We shall use the same notation, for this P, as in 2.2: $\tilde{\mathfrak{G}}$; $\tilde{K}_1, \tilde{K}_2, \cdots, \tilde{K}_{\tilde{k}}$; $K_1, K_2, \cdots, K_{\tilde{k}}$; $\tilde{Z}$; $\tilde{B}^{\Pi_f}_{(\delta)}$ and etc. Since each $\mathfrak{N}$-block $\mathfrak{B}^{\mathfrak{N}}_\sigma$ of $\mathfrak{G}$ is a collection of Π_f-blocks $B^{\Pi_f}_\delta$ of $\mathfrak{G}$, we may define the collection $\tilde{\mathfrak{B}}^{\mathfrak{N}}_{(\sigma)}$ of Π_f blocks $\tilde{B}^{\Pi_f}_{(\delta)}$ of $\tilde{\mathfrak{G}}$ such that each $\tilde{B}^{\Pi_f}_{(\delta)}$ is contained in a $B^{\Pi_f}_\delta$ with $B^{\Pi_f}_\delta \subseteq \mathfrak{B}^{\mathfrak{N}}_\sigma$. It is easy to see that each $\tilde{\mathfrak{B}}^{\mathfrak{N}}_{(\sigma)}$ is a collection of $\tilde{\mathfrak{N}}$-blocks $\mathfrak{B}^{\tilde{\mathfrak{N}}}_{\tau}$ of $\tilde{\mathfrak{G}}$, where $\tilde{\mathfrak{N}} = \mathfrak{N} \cap \tilde{\mathfrak{G}}$. We denote by $\tilde{\Delta}_{(\sigma)}$ the idempotent of $\tilde{Z}$ associated with $\tilde{\mathfrak{B}}^{\mathfrak{N}}_{(\sigma)}$:

$$\tilde{\Delta}_{(\sigma)} = \sum_{\tilde{B}^{\Pi_f}_{(\delta)} \subseteq \tilde{\mathfrak{B}}^{\mathfrak{N}}_{(\sigma)}} \tilde{E}^{\Pi_f}_{(\delta)}.$$

Then, by [2. 2. A] and (5), we obtain

[3. 2. A] *For $\mu = 1, 2, \cdots, \tilde{k}$, we have*

$$K_\mu \Delta^{\mathfrak{N}}_\sigma = \sum_{\nu=1}^{\tilde{k}} a^{\mathfrak{N}}_{\sigma,\mu\nu} K_\nu$$

and

$$\tilde{K}_\mu \tilde{\Delta}_{(\sigma)} = \sum_{\nu=1}^{\tilde{k}} a^{\mathfrak{N}}_{\sigma,\mu\nu} \tilde{K}_\nu$$

with the same coefficients $a^{\mathfrak{N}}_{\sigma,\mu\nu}$.

We shall say that two elements L and M of the Π_f-section $\mathfrak{S}^{\Pi}_f(P)$ of P in $\mathfrak{G}$ belong to a same Π_f *subsection* of P in $\mathfrak{G}$ with regard to $\mathfrak{N}$ if and only if the following two conditions are satisfied:

(a) For any normal subgroup $\mathfrak{M}$ of $\mathfrak{G}$ which contains $\mathfrak{N}$, "$L \in \mathfrak{M}$" is equivalent to "$M \in \mathfrak{M}$".

b For any normal subgroup $\tilde{\mathfrak{M}}$ of $\tilde{\mathfrak{G}}$ which contains $\mathfrak{N}\cap\tilde{\mathfrak{G}}$, "$Q\in\tilde{\mathfrak{M}}$" is equivalent to "$R\in\tilde{\mathfrak{M}}$", where Q and R are two Π_f-regular elements of $\tilde{\mathfrak{G}}$ such that L and M are conjugate in $\mathfrak{G}$ to PQ and PR, respectively.

Considering this construction for each $P=P_\gamma^{\Pi_f}$, we can distribute the elements of $\mathfrak{G}$ into a certain number of Π_f-subsections with regard to $\mathfrak{N}$. According to [3.2.A], we can refine [3.1.D] as follows:

[3.2.B] *If L and M are two elements of $\mathfrak{G}$ which belong to different Π_f-subsections of $\mathfrak{G}$ with regard to $\mathfrak{N}$, then*

$$\sum_{\chi_i\in\mathfrak{B}}\chi_i(L)\chi_i(M^{-1})=0$$

for each $\mathfrak{N}$-block $\mathfrak{B}=\mathfrak{B}_\sigma^{\mathfrak{N}}$ of $\mathfrak{G}$.

[3.2.C] *If χ_i and χ_j are two irreducible characters of $\mathfrak{G}$ which belong to different $\mathfrak{N}$-blocks of $\mathfrak{G}$, then*

$$\sum_{G\in\mathfrak{S}}\chi_i(G)\chi_j(G^{-1})=0$$

for each Π_f-subsection $\mathfrak{S}$ of $\mathfrak{G}$ with regard to $\mathfrak{N}$.

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ON THE THEORY OF DIFFERENTIAL EQUATIONS IN COORDINATED SPACES

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(Received May 31, 1961)

1 *Preliminary.* Most of ordinary differential equations of infinite dimension have been treated, roughly speaking, in two manners. One manner is mainly analytical as treated by K. P. Persidskii [1, 2](1) and other Soviet mathematicians in cases of denumerable systems of differential equations. The other is rather topologico-analytical; namely, there have been investigated differential equations in Banach spaces [for example, Massera, 3]. Investigations in the latter manner may seem apparently to be more general than those of the Soviet mathematicians, but these investigations are concerned mostly, to translate in the terms of topology, with differential equations in the space (m) or in the linear space (s) with the product topology, the latter of which is not a Banach space. Whereas, on the other hand, they are explicitely analytical and somewhat inexact, or ambiguous, to consider from the topological standpoint. To avoid these inexactnesses or ambiguities, we have investigated differential equations in coordinated spaces as a generalization of denumerable systems of differential equations [4, 5]. Namely, given an infinite system of differential equations

$$\frac{dx_i}{dt}=f_i(t, x_1, x_2, \cdots, x_n, \cdots) \tag{1}$$

$$(i=1, 2, \cdots, n, \cdots),$$

we consider this as a differential equation in a linear space E, and denote it by

$$\frac{dx}{dt}=f(t, x), \tag{2}$$

where x is a sequence $\{x_1, x_2, \cdots, x_n, \cdots\}$ of real numbers x_n, and called a *point* of E.

Here we must give some basic definitions and notations. A coordinated space E is a sequence space, whose element $x=\{x_1, x_2, \cdots, x_n, \cdots\}$, or abbreviated $\{x_n; n=1, 2, \cdots\}$, is a sequence of real numbers x_n and whose topology is locally convex and such that each mapping $x_n(x)$: $x\rightarrow x_n (n=1, 2, \cdots)$ is linear and continuous. For any point $x=\{x_1, x_2, \cdots, x_n, \cdots\}$, the point, which is constructed by equating some of the coordinates of x to zero, is called a *projection* of the point x, and especially that which is constructed by equating the coordinates with indices greater than n to zero is called a *section* of x and denoted by $x^{[n]}$. A coordinated space is called to have the property (P), if, for every neighborhood U of the fundamental system of neighborhoods of the origin $\mathfrak{U}$, the relation $x\in U$ implies that every projection of x also belongs to U. The space E is called to have the property "*Abschnittskonvergenz*" or simply (AK) [5, 6], if for

(1) Numbers in brackets refer to the references at the end of the paper.

every point x, the corresponding sequence of sections $\{x^{[n]}; n=1, 2, \cdots\}$ converges to the point x. Most of linear spaces used in practice have the property (P), but the property (AK) except the space (m), the space of all bounded sequences with the norm $\|x\|=\sup_n |x_n|$ and the space (c), the space of all convergent sequences with the same norm.

Let $x(t)$ be a vector function defined on the real interval I, that is, a function on I to the coordinated space E, and $x_n(t)$ its coordinate functions:

$$x(t)=\{x_1(t), x_2(t), \cdots, x_n(t), \cdots\}.$$

If the space is provided with the product topology, the limiting, accordingly the differentiation and the integration of the vector function $x(t)$ are equivalent to those of each coordinate function $x_n(t)$ respectively. Most of differential equations (1) treated by Persidskii and other Soviet mathematicians are mostly, in reality, differential equations (2) in coordinated spaces with the product topology. The present paper is concerned with differential equations in more general coordinated spaces.

We now consider the differential equation

$$\frac{dx}{dt}=f(t, x) \tag{2}$$

in the coordinated space E, where t is a real variable on the interval $I=[t_1, t_2]$ or $[t_1, \infty)$, x is a vector variable in the domain D in the space E, and $f(t, x)$ is a function on $I\times D$ to E. In paragraph 2, our main attention will be directed to the problem of existence, uniqueness and dependency of solutions of the differential equation (2) in the space E with different topologies rather than that in the space E with a fixed topology. In paragraph 3, we shall be concerned with linear, homogeneous differential equations, contrasting them with the case of finite dimension. In paragraph 4, we shall be concerned with relations between the stability of solutions of the differential equations (1) and that of solutions of the truncated differential equations associated with (2);

$$\frac{dx}{dt}=\hat{f}_{[n]}(t, x), \tag{3}$$

$$\frac{dx}{dt}=\tilde{f}_{[n]}(t, x), \tag{4}$$

$$\frac{dx}{dt}=\tilde{\tilde{f}}_{[n]}(t, x), \tag{5}$$

where the functions $\hat{f}_{[n]}(t, x)$, $\tilde{f}_{[n]}(t, x)$, $\tilde{\tilde{f}}_{[n]}(t, x)$ denote the trancated functions $f^{[n]}(t,x)$, $f(t, x^{[n]})$, $f^{[n]}(t, x^{[n]})$ respectively.

2. *Existence, Uniqueness and Dependency.* We consider the differential equation

$$\frac{dx}{dt}=f(t, x) \tag{2}$$

in a complete coordinated space E, where t is a real variable on the interval $I=[0, T]$ or $[0, \infty)$, x is a vector variable in the domain D in the space E, and $f(t, x)$ is a function on $I\times D$ to E. We suppose in the following that $f(t, x)$ is continuous.

Peano's existence theorem, i. e. the theorem of existence of solution of the equation (2) under the condition of continuity (and boundedness) alone of the function $f(t, x)$ cannot always be valid as was shown by J. Dieudonné [7]. However, we showed that Peano's existence theorem holds in Montel spaces and in their product spaces [4]. As its special case, we state the following theorem.

Theorem 1. Let E be the space (s) with the product topology, and let the function $f(t, x)$ be continuous and bounded in the domain $I\times E$. Then there exists at least one solution of the differential equation (2), which satisfies the initial condition: $x=x^0$ for $t=t^0$ and is defined in the interval I. This solution we shall call a solution through the point $(t_0, x^0)\in I\times E$, and denote by $x(t; t_0, x^0)$.

In other spaces, it will be appropriate to introduce some condition, say, Lipschitz condition to be imposed on the function $f(t, x)$. Lipschitz condition in a linear topological space E is stated as follows. For any neighborhood U of the fundamental system of neighborhoods of the origin $\mathfrak{U}$, the relation $x'-x''\in U$ implies that $f(t, x')-f(t, x'')\in kU$, where k is a positive constant, or equivalently, the relations $\|x'-x''\|_U\leqq 1$ implies that $\|f(t, x')-f(t, x'')\|_U\leqq k$, where $\|\ \ \|_U$ denotes the semi-norm defined by the neighborhood U. Or, if the space happens to be normed, Lipschitz condition is formulated as used usually:

$$\|f(t, x')-f(t, x'')\|\leqq k\|x'-x''\|.$$

The constant k may be replaced by a positive integrable function $k(t)$ of t, such that $\int_0^T k(t)\,dt<\infty$ (or $\int_0^\infty k(t)\,dt<\infty$).

The space E, as linear space, can be topologized in several manners. Of two topologies $\mathcal{J}_1$ and $\mathcal{J}_2$, let $\mathcal{J}_1$ be stronger (or finer) than $\mathcal{J}_2$ (or equivalently $\mathcal{J}_2$ weaker than $\mathcal{J}_1$). Since the continuity (and the convergence) of a function on I to E for the stronger topology $\mathcal{J}_1$ implies the continuity (and the convergence) for the topology $\mathcal{J}_2$, we have the following theorem.

Theorem 2. If a continuous function $x(t)$ is a solution of the differential equation (2) for a topology $\mathcal{J}_1$, then it is also a solution for a topology $\mathcal{J}_2$ weaker than $\mathcal{J}_1$.

Since, for any coordinated space, the product topology is the weakest topology, we have immediately the following corollary.

Corollary. A solution of the differential equation (2) in any coordinated space E is also a solution of the differential equation (2) for the product topology.

The converse is not always true, as is shown by the following example, given by K. P. Persidskii [1].

Example 1. We consider the differential equation

$$\frac{dx}{dt}=f(t, x),$$

where

$$f_n(t, x) = -x_n + x_{n+1}.$$

Let $\varphi(t)$ be a function defined as follows:

$$\begin{aligned} \varphi(t) &= 0 \qquad \text{for } t=0, \\ &= e^{-\frac{1}{t^2}} \qquad \text{for } t \neq 0. \end{aligned}$$

Then the vector function $x(t) = \{x_n(t)\,;\, n=1, 2, \cdots\}$, where $x_n(t) = e^{-t} \dfrac{d^{n-1}}{dt^{n-1}} \varphi(t)$, is a solution of the given equation for the product topology. But this function $x(t)$ does not exist for another topology, for example, for the topology of the space (m) or the space $(l^p)(1 \leqq p < \infty)$.

The converse, however, is true in the sense of the following theorem.

Theorem 3. Let the function $f(t, x)$ be continuous and bounded in the domain $I \times D \subset R^1 \times E$ for a topology $\mathcal{T}_1$. Let $x(t)$ be a solution through the point (t_0, x^0) of the differential equation (2) for another topology $\mathcal{T}_2$ weaker than $\mathcal{T}_1$. If the function $x(t)$ is a continuous function on a subinterval I' containing t_0 to the space E for the topology $\mathcal{T}_1$, then it is a solution, in the interval I', of the differential equation (2) also for this topology.

PROOF. By virtue of the well-known mean value theorem, we have, for the topology $\mathcal{T}_2$,

$$\frac{x(t+h) - x(t)}{h} = x'(t+\theta h) \qquad (0 < \theta < 1)$$

for t and $t+h \in I'$. Since $x(t)$ is a solution of the differential equation (2), we have

$$x'(t+\theta h) = f(t+\theta h, x(t+\theta h)),$$

and therefore the equality

$$\begin{aligned} &\frac{x(t+h) - x(t)}{h} - f(t, x(t)) \\ &\qquad = f(t+\theta h, x(t+\theta h)) - f(t, x(t)) \end{aligned}$$

and the continuity of $x(t)$ and $f(t, x)$ for the topology $\mathcal{T}_1$ imply the relation:

$$\lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = f(t, x(t))$$

for the topology $\mathcal{T}_1$, which proves the theorem.

Corollary. Let the function $f(t, x)$ be continuous and bounded in the domain $I \times D \subset R^1 \times E$ for a topology $\mathcal{T}$. Let $x(t)$ be a solution through the point (t_0, x^0) of the differential equation (2) for the product topology. If the function $x(t)$ is a continuous function on a subinterval I' containing t_0 to the space E for the topology $\mathcal{T}$, then it is a solution, in the interval I', of the differential equation (2) also for this topology.

Citing example 1, the function $x(t)$, which is a solution for the product topology, is neither a solution for the (l^2) topology nor for the (m) topology, because it is not a

continuous function for either topology. On the other hand, the function $\bar{x}(t)=\left\{e^{-t}\times \sum_{i=0}^{\infty} x_{n+i}^0 \frac{t^i}{i!}\ ;\ n=1, 2, \cdots\cdots\right\}$ is evidently a solution through the point (t_0, x^0) for the product topology, where $x^0=\{x_1^0, x_2^0, \cdots, x_n^0, \cdots\}$. In case x^0 is a point of the space (m), the function $\bar{x}(t)$ is a continuous function also for the (m) topology. In fact, at first, we have

$$\begin{aligned}\|\bar{x}(t)\| &= \sup_n \left| e^{-t} \sum_{i=0}^{\infty} x_{n+i}^0 \frac{t^i}{i!} \right| \\ &\leqq \sup_n \left| e^{-t} \|x^0\| \sum_{i=0}^{\infty} \frac{t^i}{i!} \right| = \|x^0\|.\end{aligned}$$

Secondly, we have

$$\begin{aligned}\|\bar{x}(t+h)-\bar{x}(t)\| &\leqq \sup_n \left\{ \left| e^{-(t+h)} \sum_{i=0}^{\infty} x_{n+i}^0 \left(\frac{(t+h)^i}{i!} - \frac{t^i}{i!} \right) \right| \right. \\ &\quad \left. + \left| (e^{-(t+h)}-e^{-t}) \sum_{i=0}^{\infty} x_{n+i}^0 \frac{t^i}{i!} \right| \right\} \\ &\leqq \|x^0\| \left\{ e^{-(t+h)} |e^{t+h}-e^t| + |e^{-(t+h)}-e^{-t}| e^t \right\},\end{aligned}$$

which shows the continuity of $\bar{x}(t)$ for the (m) topology. Therefore, in case x^0 is a point of the space (m), the function $\bar{x}(t)$ is a solution also for the (m) topology.

Further, we investigate a special case when a solution for the product topology becomes a solution for a topology $\mathcal{J}$. Namely we have the following theorem.

Theorem 4. Let E be a complete coordinated space with the properties (P) and (AK), and let $f(t, x)$ be continuous, bounded and satisfy Lipschitz condition in the domain $I\times D \subset R^1\times E$. If $x(t)$ is a continuous function such that its section $x^{[n]}(t)$ is a solution $x(t; t_0, x^{[n]})$ of the truncated differential equation

$$\frac{dx}{dt}=\tilde{\tilde{f}}_{[n]}(t,x), \tag{5}$$

associated with the differential equation (2), then $x(t)$ is a solution $x(t; t_0, x^0)$ of the equation (2).

For the proof, we propose a lemma, which we proved previously [5].

Lemma 1. Let $x(t)$ be a continuous function on a bounded interval I to a coordinated space E with the property (AK). Then the sections $x^{[n]}(t)$ converge to $x(t)$ uniformly for $t\in I$, that is, given an arbitrary neighborhood U, there exists a positive integer N (dependent on U, but) independent of t such that $x(t)-x^{[n]}(t)\in U$ for $n\geqq N$.

PROOF OF THE THEOREM. From the supposition on $x^{[n]}(t)$, it is written as follows:

$$x^{[n]}(t)=x^{0[n]}+\int_0^t f^{[n]}(t, x^{[n]}(t))dt.$$

Let $y(t)$ be defined by the relation

$$y(t)=x^0+\int_0^t f(t, x(t))dt. \tag{6}$$

Then we have

$$y(t)-x^{[n]}(t)=(x^0-x^{0[n]})+\int_0^t [f(t, x(t))-f^{[n]}(t, x(t))]dt$$
$$+\int_0^t [f^{[n]}(t, x(t))-f^{[n]}(t, x^{[n]}(t))]dt. \tag{7}$$

Let U be an arbitrarily given neighborhood of the origin in E. Then there exists a positive integer N, such that

$$x_0-x^{0[n]}\in \frac{1}{3}U \qquad \text{for } n\geqq N, \tag{8}$$

and, by virtue of the above lemma,

$$\int_0^t [f(t, x(t))-f^{[n]}(t, x(t))]dt \in \frac{1}{3}U \qquad \text{for } n\geqq N, \tag{9}$$

and lastly, by the property (P) and Lipschitz condition imposed on $f(t, x)$,

$$\int_0^t [f^{[n]}(t, x(t))-f^{[n]}(t, x^{[n]}(t))]dt \in \frac{1}{3}U \qquad \text{for } n\geqq N. \tag{10}$$

From the relations (8), (9), (10), and (7), we have

$$y(t)-x^{[n]}(t) \in U \qquad \text{for } n\geqq N,$$

whence $x^{[n]}(t)$ converges to $y(t)$ uniformly on I, and therefore $y(t)$ coincides with $x(t)$. Accordingly, from the relation (6), we have

$$x(t)=x^0+\int_0^t f(t, x(t))dt,$$

which proves the theorem.

In case E is a Banach space (that is, a normed and complete space), from the Lipschitz condition on $f(t, x)$, it follows by the method of successive approximations, as is known, Cauchy's theorem of existence and uniqueness, which, for the convenience of formulation, we add as a theorem.

Theorem 5. Let E be a Banach space, and let $f(t, x)$ be continuous, bounded and satisfy Lipschitz condition in the domain $I\times E$. Let $(t_0, x^0)\in I\times E$. Then there exists a unique solution $x(t;\, t_0, x^0)$ of the differential equation (2).

As regards Lipschitz condition, we give a remark. One cannot find a simple implication relation between Lipschitz condition for a stronger topology and that for a weaker topology. Namely, Lipschitz condition for a stronger (weaker) topology does not imply that for a weaker (stronger) topology as is shown by the following examples.

Example 2. We consider again the differential equation given in example 1, that is,

$$\frac{dx}{dt}=f(t, x),$$

where

$$f_n(t, x)=-x_n+x_{n+1}.$$

In case the space E is provided with the (m) or (l^p) topology $(1 \leqq p < \infty)$, a simple calculation shows that

$$\|f(t, x') - f(t, x'')\| \leqq 2\|x' - x''\|,$$

which is Lipschitz condition with $k=2$. Whereas, in case E is provided with the product topology, Lipschitz condition does not hold, as will be shown by theorem 8 in the next paragraph. As a matter of fact, in the space (s) with the product topology, the equation has a particular solution

$$\chi(t) = \left\{ e^{-t} \frac{d^{n-1}}{dt^{n-1}} \varphi(t)\,; n=1, 2, \cdots \right\}$$

as is already shown in example 1. Further, through the point (t_0, x^0), there exists a solution

$$\bar{x}(t) = \left\{ \sum_{i=0}^{\infty} x_{n+i}^0 \frac{t^i}{i!}\,;\ n=1, 2, \cdots \right\},$$

as is also already shown. Thus the function

$$x(t) = \bar{x}(t) + c\chi(t),$$

where c is an arbitrary constant, is also a solution through the point (t_0, x^0). Therefore, in the space (s) with the product topology, the differential equation has an infinite number of solutions $x(t;\, t_0, x^0)$ through the point (t_0, x^0), since c is arbitrary. On the contrary in the space (m) or $(l^p)\,(1 \leqq p < \infty)$, the differential equation has a unique solution $x(t;\, t_0, x^0) = \bar{x}(t)$ through the point (t_0, x^0).

Example 3.(2) We consider the differential equation

$$\frac{dx}{dt} = f(t, x)$$

where

$$\begin{aligned} f_n(t, x) &= x_1 && \text{for } n=1 \\ &= x_{n-1} && \text{for } n>1. \end{aligned}$$

In case the space E is provided with the (m) or the (l^2) topology, it is evident that

$$\|f(t, x') - f(t, x'')\| \leqq \|x' - x''\|,$$

which is Lipschitz condition with $k=1$. Whereas, in case E is provided with the direct sum topology (and in this case E is considered as the algebraic direct sum of a denumerable number of real lines $R_n (n=1, 2, \cdots)$), that is, a fundamental system of neighborhoods of the origin $\mathfrak{U}$ is given by

$$U_\varepsilon = \{x;\, |x_n| < \varepsilon_n, n=1, 2, \cdots\cdots\}$$

where $\varepsilon = \{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n, \cdots\}$ is any sequence of positive numbers, Lipschitz condition cannot always be valid, since the implication relation:

(2) This example is due to my colleague Y. Kōmura

$$|x'_{n-1}-x''_{n-1}|<\varepsilon_{n-1} \text{ implies } |x'_{n-1}-x''_{n-1}|<k\varepsilon_n$$

for some fixed positive constant k, cannot always be valid. In fact, for a sequence $\varepsilon=\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n, \cdots\}$, such that $\frac{\varepsilon_n}{\varepsilon_{n-1}}\to 0$ for $n\to\infty$, there exists no fixed positive constant k, such that the above implication relation holds. The direct sum topology is strictly stronger than the (m) and (l^2) topologies.

The situation that Lipschitz condition for a weaker topology does not imply that for a stronger topology, is immaterial for the uniqueness of solution. In fact, if the uniqueness of solution holds for a weaker topology $\mathcal{J}_2$, that is, if through the point (t_0, x^0) there exists a unique solution $x(t; t_0, x^0)$ for $\mathcal{J}_2$, then $x(t; t_0, x^0)$ will also be, if there exists any solution through the point (t_0, x^0) for a stronger topology $\mathcal{J}_1$, a unique solution through the point (t_0, x^0) for $\mathcal{J}_1$, because, by virtue of theorem 3, a solution through the point (t_0, x^0) for the topology $\mathcal{J}_1$ is also a solution through the point (t_0, x^0) for the weaker topology $\mathcal{J}_2$, which is unique by the hypothesis. This fact can be formulated as the following theorem.

Theorem 6. If there exists a unique solution $x(t; t_0, x^0)$ through the point (t_0, x^0) in the space E for a topology $\mathcal{J}_2$, then it is also, if there exists any solution through the point (t_0, x^0) for a topology $\mathcal{J}_1$ stronger than $\mathcal{J}_2$, the very unique solution through the point (t_0, x^0) for $\mathcal{J}_1$.

In a previous paper [5], we proved a theorem of continuous dependency of solutions on the function $f(t, x)$ on the right hand side of the differential equation (2). In this connection, we give here a theorem of continuous dependency on the initial condition, whose proof is quite analogous to that of the previous theorem and therefore is omitted.

Theorem 7. Let E be a complete coordinated space, and $f(t, x)$ be continuous, bounded and satisfy Lipschitz condition in the domain $I\times E$, where I here denotes a bounded interval $[0, T]$. Then, t_0 being fixed, the solution $x(t; t_0, x^0)$ through the point (t_0, x^0) depends on the initial value x^0 of x continuously, that is, to express more precisely, for an arbitrarily given neighborhood of the origin U, $x^0-x^{0'}\in U$ implies the relation:

$$x(t; t_0, x^0)-x(t; t_0, x^{0'})\in e^{\int_0^T k(t)dt}U,$$

where $k(t)$ is the Lipschitz function (or constant).

3. *Linear, homogeneous equations.* If the function $f(t, x)$ on the right hand side of the differential equation (2) satisfies the conditions:

$$f(t, x'+x'')=f(t, x')+f(t, x'') \qquad \text{for all } x', x''\in E,$$
$$f(t, \alpha x)=\alpha f(t, x) \qquad \text{for all real } \alpha \text{ and all } x\in E,$$

the function $f(t, x)$ is called *linear, homogeneous*, and the equation (2) also *linear, homogeneous.* If the function $f(t, x)$ is the sum of a linear, homogeneous function and a function of t only, then the equation (2) is called *linear, non-homogeneous.*

Let E be a coordinated space with the property (AK), and then each point x of E can be represented as follows:

$$x=\sum_{i=1}^{\infty} x_i e^i = x^{[n]} + (x - x^{[n]})$$

and

$$x^{[n]} = \sum_{i=1}^{n} x_i e^i,$$

where e^i are basis elements $\{\delta_{in};\, n=1, 2, \cdots\}$ [8. p. 189, theorem 1]. Let $f(t, x)$ be linear, homogeneous and continuous, and then

$$\begin{aligned} f(t, x) &= f(t, x^{[n]}) + f(t, x - x^{[n]}) \\ &= \sum_{i=1}^{n} x_i f(t, e^i) + f(t, x - x^{[n]}). \end{aligned}$$

For $n \to \infty$, $x - x^{[n]} \to 0$, and by virtue of the continuity of $f(t, x)$, we have $f(t, x - x^{[n]}) \to 0$. Therefore we have

$$f(t, x) = \sum_{i=1}^{\infty} x_i f(t, e^i).$$

Here let the vector $f(t, e^i)$ be denoted by

$$\{a_{i1}(t), a_{i2}(t), \cdots, a_{in}(t), \cdots\}$$

and then the vector $f(t, x)$ can be represented by as the multiplication of an infinite matrix by the column vector $'x$, the transposed of x considered as row vector.

$$f(t, x) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots\cdots & a_{1n}(t) & \cdots\cdots \\ a_{21}(t) & a_{22}(t) & \cdots\cdots & a_{2n}(t) & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1}(t) & a_{m2}(t) & \cdots\cdots & a_{mn}(t) & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{pmatrix}$$

or simply,

$$f(t, x) = A(t)\,'x,$$

where

$$A(t) = \big(a_{ij}(t)\big), \quad 'x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{pmatrix}.$$

Hence we have the following lemma.

Lemma 2. Any differential equation, which is linear, homogeneous and continuous in a coordinated space E with the property AK, can be written in the form of the multiplication of an infinite matrix by the column vector $'x$, the trnasposed of x as row vector,

$$\frac{dx}{dt} = A(t)\,'x.$$

where

$$A(t)=\left(a_{ij}(t)\right)=\begin{pmatrix} a_{11}(t)\ a_{12}(t)\ \cdots\cdots\ a_{1n}(t)\ \cdots\cdots \\ a_{21}(t)\ a_{22}(t)\ \cdots\cdots\ a_{2n}(t)\ \cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}(t)\ a_{m2}(t)\cdots\cdots a_{mn}(t)\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{pmatrix}.$$

Specially, when the coordinated space E is provided with the product topology or the direct sum topology, the following theorem can be easily verified.

Theorem 8. In order that Lipschitz condition should be satisfied for the linear, homogeneous equation

$$\frac{dx}{dt}=A(t)'x$$

for the product topology or the direct sum topology, it is necessary that the infinite matrix $A(t)$ *is of the triangular form, or more precisely,* $a_{ij}(t)=0$ *for* $i<j$ *in the case of the product topology and* $a_{ij}(t)=0$ *for* $i>j$ *in the case of the direct sum topology.*[3]

Thus the differential equation in example 1 is represented as follows:

$$\frac{dx}{dt}=A'x,$$

where

$$A=\begin{pmatrix} -1 & 1 & & & 0 \\ & -1 & 1 & & \\ & & -1 & \ddots & \\ 0 & & & \ddots & \ddots \end{pmatrix}.$$

For this differential equation, considered for the product topology, Lipschitz condition cannot hold. Since Lipschitz condition is a sufficient condition for the uniqueness of solution of the differential equation also in a coordinated space with the properties (AK) and (P), accordingly in the space with the product topology, it would not be unnatural that the above equation has an infinite number of solutions through the point (t_0, x^0) for the product topology, as is shown in example 2.

Let E be a complete coordinated space with the properties (P) and (AK). Consider the linear, homogeneous differential equation

$$\frac{dx}{dt}=A(t)'x$$

where the right hand side is continuous and satisfies Lipschitz condition. When t_0

(3) In the present paper, for the definiteness, as the fundamental system of neighborhoods of the origin of the product topology and that of the direct sum topology we take the family of the sets $U_{n,m}=\{x;\ \sup_{i\leq n}|x_i|\leq 1/m\}$, $(n,m=1,2,\cdots\cdots)$ and that of the sets $U_\varepsilon=\{x;\ |x_n|<\varepsilon_n,\ n=1,2,\cdots\cdots\}$, where $\varepsilon=\{\varepsilon_n\}$ is any decreasing sequence of positive numbers ε_n, respectively, unless otherwise stated.

being fixed, by virtue of the property of the uniqueness of solution, the totality of the solutions $x(t; t_0, x^0)$ forms a linear space, denoted by $\mathcal{E}$, which is algebraically isomorphic to the original space E, what can be verified analogously to the case of a finite-dimensional space. We shall show that this isomorphism turns to be also topological.

Let $x^i(t)$ denote the solution $x(t; t_0, e^i)$, that is, the solution passing through the point (t_0, e^i). Then the solution $x(t; t_0, x^{0[n]})$ is represented as follows:

$$x(t; t_0, x^{0[n]}) = x\left(t; t_0, \sum_{i=1}^{n} x_i^0 e^i\right) = \sum_{n=1}^{n} x_i^0 x^i(t).$$

For an arbitrarily given neighborhood of the origin U, there exists, by virtue of the property (AK), a positive integer N, such that $n \geqq N$ implies $x^0 - x^{0[n]} \in U$, and therefore, by virtue of theorem 7

$$x(t; t_0, x^0) - x(t; t_0, x^{0[n]}) \in e^{\int_0^T k(t)\, dt}\, U,$$

where $k(t)$ is the Lipschitz function (or constant). This shows that the solution $x(t; t_0, x^{0[n]})$ converges to the solution $x(t; t_0, x^0)$ and that uniformly on the interval $[0, T]$:

$$\begin{aligned} x(t; t_0, x^0) &= \lim_{n\to\infty} x(t; t_0, x^{0[n]}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} x_i^0 x^i(t) = \sum_{i=1}^{\infty} x_i^0 x^i(t). \end{aligned}$$

This fact shows that the set $\{x^i(t); i=1, 2, \cdots\}$ forms an algebraic basis of the derived linear space $\mathcal{E}$. Moreover, the mapping: $x^0 \to x(t; t_0, x^0)$ is not only one-to-one, but, if $\mathcal{E}$ is topologized by the uniform convergence, also continuous and inversely continuous, as is easily seen. Hence we obtain the following theorem.

Theorem 9. Let E be a complete coordinated space with the properties (P) and (AK). Let the right hand side of the linear, homogeneous differential equation

$$\frac{dx}{dt} = A(t)'x \tag{11}$$

be continuous and satisfy Lipschitz condition in the domain $I \times E \subset R^1 \times E$. Then, t being fixed, the totality of the solutions $x(t; t_0, x^0)$ forms a linear topological space $\mathcal{E}$ with the topology of uniform convergence, which is topologically isomorphic to the space E, and the solution $x(t; t_0, x^0)$ is represented in the form:

$$x(t; t_0, x^0) = \sum_{i=1}^{\infty} x_i^0 x^i(t), \tag{12}$$

where $x^i(t) = x(t; t_0, e^i)$ $(i=1, 2, \cdots)$.

The set of the solutions $\{x^i(t); i=1,2, \cdots\}$ is called the *fundamental system of solutions* of the linear, homogenuous differential equation (11). Varying the initial value x^0, we shall call the solution of the form (12), the *general solution* of the differential equation (11).

We cite again the linear, homogenuous differential equation in example 3:

$$\frac{dx}{dt} = A'x \tag{13}$$

where

$$A=\begin{pmatrix} -1 & 1 & & & 0 \\ & -1 & 1 & & \\ & & -1 & 1 & \\ 0 & & & \ddots & \ddots \end{pmatrix}.$$

For example, let E be provided with the (l^2) topology, and then the conditions are evidently satisfied. Thus we have the fundamental system of solutions:

$$x^1(t)=e^{-t}\{1, 0, 0, \cdots\cdots, 0, \cdots\cdots\},$$
$$x^2(t)=e^{-t}\left\{\frac{t}{1!}, 1, 0, \cdots\cdots, 0, \cdots\cdots\right\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x^n(t)=e^{-t}\left\{\frac{t^{n-1}}{(n-1)!}, \frac{t^{n-2}}{(n-2)!}, \frac{t^{n-3}}{(n-3)!}, \cdots\cdots, 1, 0, \cdots\cdots\right\}.$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Therefore, if x^0 is a point of the space (l^2), for the solution $x(t; t_0, x^0)$ of the differential equation (13), we have the representation:

$$x(t; t_0, x^0)=\sum_{i=1}^{\infty} x_i^0 x^i(t). \tag{14}$$

This representation can be rewritten also in the form of coordinate functions as follows:

$$x(t; t_0, x^0)=\{x_1(t), x_2(t), \cdots, x_n(t), \cdots\}, \tag{15}$$

where

$$x_n(t)=\sum_{i=1}^{\infty} x_{n+i}^0 \frac{t^i}{i!} \qquad (n=1, 2, \cdots).$$

If, however, E is provided with the (m) topology, then an immediate application of the theorem is clearly impossible, since the property (AK) is not satisfied in this case. Hence the representation (14) of solution is not valid. But the second representation (15) is valid. In fact, the vector function on the right hand side of (15) is a solution of the differential equation (13) passing through the point (t_0, x^0) for the product topology, and besides continuous for the (m) topology; therefore this vector function (15) is a solution through the point (t_0, x^0), by virtue of the corollary of theorem 3, for the (m) topology. Thus the two representations (14) and (15) are not generally equivalent.

To dissolve this inharmony between these two representations of solution, we introduce another notion of (AK); for example, the space (m) possesses the property (AK) for the $\sigma(m, l)$ topology, that is, the property $\sigma(m, l)$-(AK) [6, p. 151, theorem 2]. Let the sum of a series $\sum$ convergent for the $\sigma(m, l)$ topology be denoted by $\sigma(m, l)$-$\sum$. Then it follows easily that

$$x(t; t_0, x^0)=\sigma(m, l)\text{-}\sum x_i^0 x^i(t),$$

that is, the representation (14) is valid for the $\sigma(m, l)$ topology.

This situation can be generalized as follows. Let $\mathcal{J}$ be the proper topology of a complete coordinated space E, and $\mathcal{J}'$ be another topology on E weaker than $\mathcal{J}$. To distinguish the properties P and AK, convergence, limit, and sum $\sum$ for the topology $\mathcal{J}'$ from those for the topology $\mathcal{J}$ respectively, we denote the former by $\mathcal{J}'$-(P) and $\mathcal{J}'$-(AK), $\mathcal{J}'$-convergence, $\mathcal{J}'$-limit, and $\mathcal{J}'$-$\sum$ respectively. Then we shall easily obtain the following theorem.

Theorem 10. Let E be a complete coordinated space with the topology $\mathcal{J}$, and possess the properties $\mathcal{J}'$-P and $\mathcal{J}'$-AK, where the topology $\mathcal{J}'$ is weaker than $\mathcal{J}$. Let the right hand side of the linear, homogeneous differential equation

$$\frac{dx}{dt} = A(t)'x$$

be continuous and satisfy Lipschitz condition in the domain $I \times E \subseteq R^1 \times E$. Then, t_0 being fixed, the totality of the solutions $x(t; t_0, x^0)$ through the point (t_0, x^0) forms a linear topological space $\mathfrak{S}$ with the topology of uniform $\mathcal{J}'$-convergence, which is topologically isomorphic to the space E, and the solution $x(t; t_0, x^0)$ is represented as follows:

$$x(t; t_0, x^0) = \mathcal{J}'\text{-}\sum_{i=1}^{\infty} x_i^0 x^i(t),$$

where $x^i(t) = x(t; t_0, e^i)$ $(i = 1, 2, \cdots)$.

4. *Stability.* In this paragraph, we assume that $f(t, 0) = 0$ identically for t, that is, the differential equation in the linear topological space E

$$\frac{dx}{dt} = f(t, x) \tag{2}$$

has the trivial solution $x=0$. Analogously to the case of finite-dimensional spaces, we define the stability in the sense of Liapounov as follows: the trivial solution $x=0$ of the differential equation (2) is *stable*, if, for an arbitrary pair of a neighbourhood of the origin U and a value $t_0(\geqq 0)$ of t, there exist a value T of t and a neighborhood of the origin V, such that $x^0 \in V$ and $t \geqq T$ imply $x(t; t_0, x^0) \in U$, and *unstable* otherwise. The trivial solution $x=0$ is called to be *asymptotically stable*, if it is stable and if, moreover, for any neighborhood of the origin U, there exists a T_U of t, such that $t \geqq T_U$ implies $x(t; t_0, x^0) \in U$. In particular, when the space E is coordinated, furthermore we define another notion of stability as follows; the trivial solution $x=0$ of the equation (2) is *coordinate-wise stable*, if, for an arbitrary triple of coordinate number n, a positive number ε and a value $t_0(\geqq 0)$ of t, there exist a value T of t and a neighborhood of the origin V, such that $x^0 \in V$ and $t \geqq T$ imply $|x_n(t; t_0, x^0)| < \varepsilon$, and *coordinate-wise unstable* otherwise. The asymptotic coordinate-wise stability is defined similarly. To see from our standpoint, most Soviet mathematicians cited above treated, in reality, the stability for the m topology and the asymptotic coordinate-wise stability. In the following, we shall first analyze some relations between stabilities, that is, some implication relations.

In the coordinated space E, since the mapping $x \to x_n$ is continuous, it follows immediately the following theorem.

Theorem 11. In the coordinated space, the (asymptotic) stability implies the (asymptotic) coordinate-wise stability.

The converse, however, is not true, as is shown by the following example.

Example 4. In the space (l^2), we consider the differentiable equation

$$\frac{dx}{dt} = f(x),$$

where

$$\begin{aligned} f_n(x) &= -\lambda x_1 && \text{for } n=1, \\ &= x_{n-1} - \lambda x_n && \text{for } n>1, \end{aligned}$$

and λ is a positive number such that $0<\lambda<\frac{1}{4}$. The function $f(x)$ is bounded and satisfies Lipschitz condition, accordingly is continuous. In fact, a simple calculation shows that

$$\|f(x)\| \leqq (\lambda+1)\|x\|,$$

and consequently

$$\|f(x') - f(x'')\| \leqq (\lambda+1)\|x' - x''\|.$$

The fundamental system of solutions of (2) is given by

$$x^1(t) = e^{-\lambda t}\left(1, \frac{t}{1!}, \frac{t^2}{2!}, \cdots, \frac{t^{n-1}}{(n-1)!}, \cdots\right),$$

$$x^2(t) = e^{-\lambda t}\left(0, 1, \frac{t}{1!}, \cdots, \frac{t^{n-2}}{(n-2)!}, \cdots\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x^n(t) = e^{-\lambda t}\left(0, 0, 0, \cdots, 1, \frac{t}{1!}, \cdots\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where $\|x^n(t)\| = e^{-\lambda t}\sqrt{\sum_{i=1}^{\infty}\left(\frac{t^{i-1}}{(i-1)!}\right)^2}$ for all n.

Accordingly, the solution $x(t; t_0, x^0)$ of (2) passing through the point $(0, x^0)$ is given by

$$x(t;\ 0,\ x^0) = x_1^0 x^1(t) + x_2^0 x^2(t) + \cdots + x_n^0 x^n(t) + \cdots,$$

or

$$= \{x_1(t), x_2(t), \cdots, x_n(t), \cdots\},$$

where

$$x_1(t) = x_1^0 e^{-\lambda t},$$

$$x_2(t) = \left(x_1^0 \frac{t}{1!} + x_2^0\right) e^{-\lambda t},$$

$$x_3(t)=\left(x_1^0\frac{t^2}{2!}+x_2^0\frac{t}{1!}+x_3^0\right)e^{-\lambda t},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_n(t)=\left(x_1^0\frac{t^{n-1}}{(n-1)!}+x_2^0\frac{t^{n-2}}{(n-2)!}+\cdots+x_n^0\right)e^{-\lambda t},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The trivial solution $x=0$ is evidently asymptically coordinate-wise stable. However, it is not stable. In fact, let $x^0=\left\{\varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon}{2^2}, \cdots, \frac{\varepsilon}{2^{n-1}} \cdots\right\}$, and then

$$\|x^0\|=\frac{2}{\sqrt{3}}\varepsilon$$

$$\|x(t)\|^2=\sum_{n=1}^{\infty}x_n^2(t)$$

$$>\varepsilon^2 e^{-2\lambda t}\left[1+\frac{t}{2}\frac{1}{1!}+\left(\frac{t}{2}\right)^2\frac{1}{2!}+\cdots+\left(\frac{t}{2}\right)^{n-1}\frac{1}{(n-1)!}+\cdots\right]$$

$$=\varepsilon^2 e^{-2\lambda t}e^{\frac{t}{2}}=\varepsilon^2 e^{\left(\frac{1}{2}-2\lambda\right)t}.$$

Consequently, for an arbitrarily small $\varepsilon, t\to\infty$ implies $\|x(t)\|\to\infty$, which shows the instability of the trivial solution $x=0$.

The general solution of the above equation converges to the trivial solution $x=0$ also for the product topology, which indicates that the (asymptotic) stability for the product topology does not imply the (asymptotic) stability for the (l^2) topology. On the other hand, that the stability for the (l^2) topology does not imply the stability for the product topology, is shown by the following example.

Example 5. Let the function $\varphi(r)$ be defined as follows:

$$\varphi(r)=(r-1) \qquad \text{for } 0\leqq r\leqq 1,$$

$$=\frac{r-1}{r} \qquad \text{for } \quad r\geqq 1.$$

In the space (l^2) we consider the equation

$$\frac{dx}{dt}=\varphi(\|x\|)x.$$

Lipschitz condition is easily verified to be satisfied:

$$\|\varphi(\|x'\|)x'-\varphi(\|x''\|)x''\|\leqq 3\rho\|x'-x''\|$$

in the sphere $\|x\|<\rho$. Hence the solution $x(t)=x(t;0,x^0)$ passing through the point $(0, x^0)$ is given by

$$\frac{x_1(t)}{x_1^0}=\frac{x_2(t)}{x_2^0}=\cdots=\frac{x_n(t)}{x_n^0}=\cdots.$$

and

$$\|x(t)\| = \frac{1}{1+\left(\frac{1}{\|x^0\|}-1\right)e^t} \qquad \text{for} \quad \|x^0\|<1,$$

$$=1 \qquad \text{for} \quad \|x^0\|=1,$$

$$=1+(\|x^0\|-1)e^t \qquad \text{for} \quad \|x^0\|>1.$$

The trivial solution $x=0$ is asymptotically stable for the (l^2) topology, but unstable for the product topology.

The above two examples show that it is impossible to find a simple implication relation between the stability for a stronger topology and that for a weaker topology. Furthermore, the above last example shows also that the stability for the product topology does not coincide with the coordinate-wise stability. In fact, the trivial solution $x=0$ of the above last example is unstable for the product topology, but stable for the (l^2) topology, accordingly by virtue of theorem 11, coordinate-wise stable. On the other hand, by virtue of the same theorem, the stability for the product topology implies the coordinate-wise stability.

Next, we shall clarify the relation between the stability of the trivial solution $x=0$ of the differential equation (2) and that of the truncated differential equations associated with (2). Here we consider the truncated differential equation

$$\frac{dx}{dt} = \tilde{\tilde{f}}_{[n]}(t, x) \tag{5}$$

only, since similar arguments are available for other truncated differential equations. If $x=0$ is a trivial solution of the differential equation (2), then it is clearly also that of the truncated differential equation (5). It follows immediately that the stability of the trivial solution $x=0$ of the equation (5) for all n does not imply that of the original equation (2). In fact, we cite again example 4, where the trivial solution $x=0$ is unstable for the (l^2) topology as was already verified. Whereas, the solution $\tilde{\tilde{x}}_{[n]}(t;\, t_0, x^0)$ through the point (t_0, x^0) of the truncated equation is

$$\tilde{\tilde{x}}_{[n]}(t;\, t_0, x^0) = \{x_1(t), x_2(t), \cdots, x_n(t), x^0_{n+1}, x^0_{n+2}, \cdots\},$$

where $x_1(t), x_2(t), \cdots, x_n(t)$ are such coordinate functions as are shown in the example. This shows the stability of the trivial solution $x=0$ of the truncated equation for all n.

We now introduce a new notion on stability in order to state a condition for that the stability of solutions of the truncated equations (5) should imply that of the original equation (2). We call the trivial solutions $x=0$ of the truncated equations (5) *stable equally for n*, or simply *equi-stable*, if, for an arbitrary pair of a neighborhorhood of the origin U and a value t_0 of t, there exist a value T of t and a neighborhood of the origin V, both independent of n, such that $x^0 \in V$ and $t \geqq T$ imply $\tilde{\tilde{x}}_{[n]}(t;\, t_0, x^0) \in U$. We obtain the following theorom.

Theorem 12. Let E be a complete coordinated space with the properties (P) and (AK). Let the right hand side $f(t, x)$ of the differential equation

$$\frac{dx}{dt} = f(t, x) \tag{2}$$

be continuous and bounded, and satisfy Lipschitz condition, in each domain $I \times D \subset R^1 \times E$, where I is any bounded subinterval of the half line $[0, \infty)$ and D a fixed domain of E. If the trivial solutions $x=0$ of the truncated differential equations

$$\frac{dx}{dt} = \tilde{\tilde{f}}_{[n]}(t, x) \tag{5}$$

associated with (2) are equi-stable, then the trivial solution $x=0$ of the original equation (2) is also stable.

PROOF. Let U be an arbitrary neighborhood of the fundemental system of neighborhoods of the origin. Then, by the hypothesis of the equi-stability, there exist a value T of t and a neighborhood V, independent of n, such that $x^0 \in V$ and $t \geq T$ imply $\tilde{\tilde{x}}_{[n]}(t; t_0, x^0) \in U/2$ for all n. Let l be an arbitrary but fixed positive number. By virtue of the théorème de réduite [5, p. 238, theorem 2], there exists a positive integer N such that

$$x(t; t_0, x^0) - \tilde{\tilde{x}}_{[N]}(t; t_0, x^0) \in U/2 \qquad \text{for } t \in [T, T+l],$$

therefore

$$x(t; t_0, x^0) \in U \qquad\qquad \text{for } t \in [T, T+l].$$

Since l is arbitrary, this implication holds for all t of the half line $[T, \infty)$, which proves the theorem.

The condition of equi-stability for the stability of the trivial solution of the original equation (2) is only sufficient, but not necessary, which is shown by the following example.

Example 6. In the space (l), we consider the linear, homogenuous differential equation

$$\frac{dx}{dt} = A'x,$$

where A is a diagonal matrix of matrixes:

$$A = \begin{pmatrix} B & & & & \\ & B & & & \\ & & B & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \cdot \end{pmatrix}$$

and B is a $(2, 2)$–matrix of the form $\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$.

Lipschitz condition is evidently satisfied. The general solution, that is, the solution $x(t; t_0, x^0)$ through the point (t_0, x^0) is given by the coordinate functions:

$$x_{2m-1}(t) = (3x^0_{2m-1} - 2x^0_{2m})e^{-(t-t_0)} + 2(x^0_{2m} - x^0_{2m-1})e^{-2(t-t_0)},$$
$$x_{2m}(t) = (3x^0_{2m-1} - 2x^0_{2m})e^{-(t-t_0)} + 3(x^0_{2m} - x^0_{2m-1})e^{-2(t-t_0)}$$
$$(m = 1, 2, \cdots).$$

Thus the trivial solution $x=0$ of the given equation is clearly asymptotically stable.

while the trivial solution of the truncated equation is stable for n even and unstable for n odd.

Hence the condition of equi-stability of the theorem may be expected to weaken. The equi-stability above defined is called anew the *equi-stability in the strong sense*, and in case the selection of the neighborhood V in the above definition is independent (uniform) of not all, but of an infinite number of values of n, the equi-stability is called the *equi-stability in the weak sense*. Then, with a slight modification of the proof, the above theorem can be replaced by the following theorem.

Theorem 13. Assume that the supposition of the theorem 12 is fulfilled. If the trivial solutions $x=0$ *of the truncated equations (5) are equi-stable in the weak sense, then the trivial solution* $x=0$ *of the original equation (2) is stable.*

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ON THE TYPE OF THE ELECTROMAGNETIC AND STRONG MESONIC INTERACTIONS OF THE FERMIONS

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(Received Sept. 3, 1961)

SUMMARY

So far, there seem no decisive solutions eather experimentally or theoretically to the problems whether there is a Pauli type interaction in the electromagnetic interaction of the Fermion and whether there is a direct coupling between the nucleon and the pion. As for the latter, there seem to be several experimental evidences supporting the derivative coupling, but these are not yet assurable. In this report, by assuming that the wave function of the Fermion is composed of two parts with opposite parities to each other, we tried to determine the type of the interactions between Fermions and Bosons. Firstly, we assume that the wave function of the Fermion is of the type

$$\psi = \cos\beta\cdot\varphi_{+} + \sin\beta\cdot\varphi_{-},$$

where φ_{+} and φ_{-} are the constituent wave functions with opposite parities and β is the arbitrary C-number constant determining the mixing ratio of φ_{+} and φ_{-}. Then, as we can not distinguish φ_{+} and φ_{-} at all for the free particle, the interaction Lagrangian density must be so constructed that the form of it is completely symmetric concerning to φ_{+} and φ_{-}. Further, from the fact that the strong and the electromagnetic interactions must be independent of the constant β determining the mixing ratio of φ_{+} and φ_{-}, we can conclude that the Pauli term in the electromagnetic interaction and the direct coupling term in the pion-nucleon system must be excluded. Physical meaning of the wave function mentioned above and the relation to the results obtained in the previous papers are also investigated.

§1. INTRODUCTION

As is well known, the interactions between the elementary particles are classified into three groups, that is the strong, the electromagnetic and the weak interactions. It is also well known that, among these interactions, the strong and the electromagnetic interactions do conserve each of P, C and T, but the weak interaction does not conserve each of them separately. Lately, the properties of these interactions are investigated in detail by many authors and a lot of selection rules are discovered, but the reason why these differences about the interactions have to happen in nature is not yet clarified at all.

Further, when we construct the interaction Lagrangian between the given elementary particles, it is also unknown whether we can construct it singly or we must have plural forms of interactions with equal rights.

So far, we do not know whether these two problems should be treated separately or should be treated in connection with each other. In this paper, we will study about

these problems in the cases of the interactions between Fermions and Bosons.

In the ordinary formalisms, the interaction Lagrangian is so constructed as to be relativistically invariant and to conserve the charge and the statistics. In addition to these, in the case of the strong interactions, further conservation laws are required. In the case considered, the reason why there are plural interaction Lagrangian densities is that they can include the derivatives of the wave functions within themselves.

Now, let us assume that the interaction Lagrangian does not contain the derivative of the Fermion wave function and does contain the derivative of the Boson wave function up to the first. We have, in general, two forms of interactions, the direct one and the derivative one. These two interactions are theoretically equivalent in all of their aspects. Both of them are, in the ordinary theory, able to reconcile in every points, and we can not insist one of them is preferable to the other. But, experimentally, the situation seems to be little bit different. In the pion-nucleon interaction, there are several evidences in which the derivative interaction is preferable to the direct interaction. In the electromagnetic interaction, usually, the interaction Lagrangian can be obtained by the substitution

$$\partial_\mu \rightarrow \partial_\mu - ieA_\mu$$

in the particle part of the Lagrangian density, but the interaction obtained as above is the direct one in the sense mentioned above and naturally we can expect the term of the derivative interaction, which we usually call "the Pauli-type interaction (Pauli-term, in short)". In this case, the Pauli-term plays only an additional role and can not interprete the essential features of the electromagnetic phenomena by itself.

Thus, each of two interactions which are obtained on the same theoretical footing seems to play differently in the actual phenomena. In other words, we might abandon one of them in interpreting the actual experimental results. It might be very interesting to settle the selection rules to distinguish these two interactions theoretically.

As it is discovered experimentally that there are many processes in which each of P, C and T does not conserve separately, we assume that the parity conservation is destroyed in every of the weak interactions. Of course, the parity is conserved in the strong interactions. These facts seem to mean that there is a kind of conservation law which hold in the strong interaction but does not hold in the weak one. In other words, we assume that, concerning this conservation law, the strong interaction is allowed, while the weak one forbidden. Therefore, we can consider that every sort of the preliminary interactions between the elementary particles are of the nature of the parity non-conserving, and only when these interactions satisfy that kind of the conservation law, the parity happens to conserve and the interactions become strong.

These assumptions are well formulated by considering the wave function of the particle composed of the two parts having the opposite parities to each other. As we can say nothing about the parity of the particle when it is free, the assumptions mentioned above could not affect the theory of the free particle. Further, so long as we are concerned about the free particle, we can not say anything about the mixing ratio β. Next, as the strong interactions are related intimately to the production processes, if it depends on the mixing ratio β, the produced particles will have the different mixing ratios as the case may be. As the relative difference of the mixing ratios could be

detected experimentally, in order to prevent the appearance of the effects of the mixing ratio on the particle produced, it must be necessary to postulate that the strong interaction is independent of the mixing ratio, too. On the contrary, the weak interactions do not conserve the parity, which means that they can depend on the C-number constant β. It is easily found that the above considerations enables us to distinguish the strong interactions and the weak interactions through the constant β. Further, the theory developed along the line mentioned above exhibits the difference between the derivative and the direct coupling which enables us to make a distinction between these two interactions.

For the sake of simplicity, we will assume that the Fermion wave function is composed of two parts having the opposite parities. Then, the quantized wave function for the actual Fermion ψ will be written as

$$\psi = \cos\beta \cdot \varphi_+ + \sin\beta \cdot \varphi_-$$

where φ_+ and φ_- are the constituent operators corresponding to the opposite parity parts, and the arbitrary constant β determine the mixing ratio of these two constituent parts. The quantized wave functions φ_+ and φ_- satisfies the Dirac's equations

$$(\gamma_\mu \partial_\mu + \kappa)\varphi_+ = 0,$$
$$(\gamma_\mu \partial_\mu - \kappa)\varphi_- = 0,$$

respectively. Now, if we put, $\varphi_- = i\gamma_5 \varphi$, then the equation for φ will become

$$(\gamma_\mu \partial_\mu + \kappa)\varphi = 0.$$

Therefore, writing

$$\varphi_+ = \varphi, \qquad \varphi_- = i\gamma_5 \varphi,$$

we obtain, as the final form of the wave function for the Fermion,

$$\psi = (\cos\beta + i\gamma_5 \sin\beta)\varphi = e^{i\beta\gamma_5}\varphi.$$

We have obtained this form of the Fermion wave function in the previous papers[1]. We already knew that this form of the wave function does not affect the theory of free Fermion. In the foregoing papers[2], we treated the same problems as the ones in this paper on the different footing. In both cases, the results obtained are the same as they should be. We will give in § 2 the short survey of the previous papers[1] and the severals of the additional considerations concerning the generalized linearization of the Klein-Gordon equation for the Fermions. In § 3, the selection rules obtained from the considerations mentioned above will be stated. § 4 will be devoted to the discussions and the conclusions.

§ 2. THE QUANTIZED WAVE FUNCTION OF THE FERMIONS

The attempt to generalize the Dirac equation is already developed in the earlier papers[1]. Here, we will give the brief survey of it and will give the considerations about the physical meaning of the formalism.

I) LINEARIZATION OF THE KLEIN-GORDON EQUATION

In order to linearize the Klein-Gordon equation for the Dirac field ψ, $(\Box-\kappa^2)\psi=0$, assuming only the invariance under the proper Lorentz transformation, we write it in the form

$$d(\partial)\Lambda(\partial)\psi=0,$$

where

$$\Lambda(\partial)=-\{\Gamma_1\gamma_\mu\partial_\mu+\Lambda_1\kappa\},$$
$$d(\partial)=-\{\Gamma_2\gamma_\mu\partial_\mu-\Lambda_2\kappa\},$$

Γ_1, Γ_2, Λ_1^1 and Λ_2 being matrices containing the Dirac matrices. These matrices can be determined from the conditions

$$d(\partial)\Lambda(\partial)\equiv(\Box-\kappa^2),$$
$$[d(\partial),\Lambda(\partial)]_-=0$$

and are found to be

$$\Gamma_1=\Gamma_2=e^{A\gamma_5},\ \Lambda_1=e^{-B\gamma_5},\ \Lambda_2=e^{B\gamma_5} \quad\cdots\cdots (1)$$

where A, B are the C-number arbitrary constants. Then, the equation of motion and the commutation relation for the wave function ψ become

$$-\Lambda(\partial)\psi=\{e^{A\gamma_5}\gamma_\mu\partial_\mu+\kappa e^{-B\gamma_5}\}\psi=0, \quad\cdots\cdots (2)$$

$$\{\psi_\alpha(x),\bar{\psi}_\beta(x')\}_+=-i\{e^{A\gamma_5}\gamma_\mu\partial_\mu-\kappa e^{B\gamma_5}\}_{\alpha\beta}\Delta(x-x'), \quad\cdots\cdots (3)$$

where $\bar{\psi}$ is defined by the relation

$$\bar{\psi}=\psi^*\eta=\psi^+e^{\frac{1}{2}\{(B+B^*)-(A-A^*)\}\gamma_5},\ \psi^+=\psi^*\gamma_4. \quad\cdots\cdots (4)$$

The matrix η is so determined as to be non-singular and to make the Lagrangian giving the equation of motion (1),

$$\bar{L}_F=-\int\psi^*\eta(e^{A\gamma_5}\gamma_\mu\partial_\mu+\kappa e^{-B\gamma_5})\psi d^4x, \quad\cdots\cdots (5)$$

hermitic.

The Lagrangian (5) can be written in the form

$$\bar{L}_F=-\int\psi^*\gamma_4\{e^{(\alpha+\alpha^*)\gamma_5}\gamma_\mu\partial_\mu+\kappa e^{(\alpha^*-\alpha)\gamma_5}\}\psi d^4x, \quad\cdots\cdots (6)$$

where

$$\alpha=\frac{1}{2}(A+B).$$

Now, if we variate the Lagrangian (5) with ψ^+, we get

$$\{e^{(\alpha+\alpha^*)\gamma_5}\gamma_\mu\partial_\mu+\kappa e^{(\alpha^*-\alpha)\gamma_5}\}\psi=0. \qquad (7)$$

This equation is equivalent to the equation (2). Indeed, it can be obtained from the equation (2) by multiplying the factor $e^{(\alpha+\alpha^*)\gamma_5-A\gamma_5}$ from the left side. This form of the equation of motion shows us that, in the above formalism, we can choose the constant A as real and the constant B as pure imaginary without the loss of generality. Of course, this means that we must take ψ in the commutation relation as ψ' and therefore the apparent form of the commutation will take a different form to the former case. It reads

$$\{\psi_\alpha(x),\psi_\beta^+(x')\}_+=-i(e^{(\alpha+\alpha^*)\gamma_5}\gamma_\mu\partial_\mu-\kappa e^{(\alpha-\alpha^*)\gamma_5})_{\alpha\beta}\Delta(x-x'). \qquad (8)$$

Now, let us transform ψ into φ by the unitary transformation S,

$$\psi=S\varphi, \qquad (9)$$

where S is assumed to be commutable with γ_5

$$[S,\gamma_5]=0.$$

If we choose S so as to satisfy the relation

$$S^2e^{(\alpha^*-\alpha)\gamma_5}=1,$$

then

$$S=e^{\frac{1}{2}(\alpha-\alpha^*)\gamma_5} \qquad (10)$$

Define Γ_μ and Γ_5 as

$$S^{-1}\gamma_\mu S=e^{-(\alpha-\alpha^*)\gamma_5}\gamma_\mu=\Gamma_\mu,\ \gamma_5=\Gamma_5, \qquad (11)$$

then Lagrangian, the equation of motion and the commutation relation for φ read

$$\bar{L}_F=-\int\varphi^*\Gamma_4\{e^{(\alpha+\alpha^*)\gamma_5}\Gamma_\mu\partial_\mu+\kappa\}\varphi d^4x, \qquad (12)$$

$$\{e^{(\alpha+\alpha^*)\gamma_5}\Gamma_\mu\partial_\mu+\kappa\}\varphi=0, \qquad (13)$$

$$\{\varphi_\alpha(x),\bar{\varphi}_\beta(x')\}_+=-i(e^{(\alpha+\alpha^*)\gamma_5}\Gamma_\mu\partial_\mu-\kappa)_{\alpha\beta}\Delta(x-x'), \qquad (14)$$

where

$$\bar{\varphi}=\varphi^*\Gamma_4. \qquad (15)$$

These relations show that the net effects of the generalization of the Dirac equation is divided into two parts; one of them is reduced to the change of the representation of the Dirac matrix and one of them is to put the factor $e^{(\alpha+\alpha^*)\gamma_5}$ to the term containing the derivative operator. For the latter, it should be emphasized that only the real part of the constant α exhibits the essential difference to the ordinary formalism.

II) REDUCING OPERATOR

Equations (2), (7) and (13) correspond to the Dirac equation of the free Fermion. Let the wave function which satisfy the ordinary Dirac equation be φ_0. We have

$$(\gamma_\mu \partial_\mu + \kappa)\varphi_0 = 0 \qquad (16)$$

The wave function ψ in I) is connected with this φ_0 by the transformation R as

$$\psi = R\varphi_0, \quad R = e^{\alpha\gamma_5} \qquad (17)$$

Formally, this transformation R is not necessary unitary. Indeed, as we only assume that the wave function of the actual Fermion is composed of two parts with opposite parities, the only operator that has the clear physical meaning is the wave function ψ and not the wave function φ_0. The discussions in the earlier papers[1),2)] are developed along this stand point of view. We called the operator R as the "*Reducing Operator*" because it reduces the formalism formally to the ordinary one.

As is stated in §1, however, our present stand points are different from the previous one. We assume that the constituent parts of the Fermion wave function have their own physical meaning, that is, they are operators corresponding to the pure parity particles. Concerning these interpretations, we will discuss in detail in subsection III). Under these situations, the transformation operator R must be unitary, which means that the constant α must be pure imaginary. In the following, we will put

$$\alpha = i\beta$$

(β: arbitrary C-number real constant),

then

$$\psi = R\varphi_0, \quad R = e^{i\beta\gamma_5}, \qquad (18)$$

and the equations (13)……(15) become

$$\bar{L}_F = -\int \bar{\varphi}(\Gamma_\mu \partial_\mu + \kappa)\varphi d^4x, \qquad (19)$$

$$(\Gamma_\mu \partial_\mu + \kappa)\varphi = 0, \qquad (20)$$

$$\{\varphi_\alpha(x), \bar{\varphi}_\beta(x')\}_+ = -i(\Gamma_\mu \partial_\mu - \kappa)_{\alpha\beta} \Delta(x - x'), \qquad (21)$$

$$\bar{\varphi} = \varphi^* \Gamma_4, \qquad (22)$$

$$\Gamma_\mu = e^{-2i\beta\gamma_5}\gamma_\mu, \quad \Gamma_5 = \gamma_5 \qquad (23)$$

As was stated previously, one of the essential differences between the ordinary formalism and the one using the generalized Dirac equation comes from the real part of the arbitrary constant α. If we require, however, the unitarity of the reducing operator R as was stated above, this part of the differences does vanish. Therefore, as for the free particle, even if we generalize the linearization of the Klein-Gordon equation assuming only the invariance under the proper Lorentz transformation, the result obtained

still keeps the invariance under the improper Lorentz transformation as it should be. To generalize the Dirac equation is thus equivalent to choose the different representation for the Dirac matrix. It must be noted that the discussions mentioned above are based on the assumption that the reducing operator must be unitary.

III) INTERPRETATION OF THE RESULT OBTAINED

As was stated above, we can obtain the family of the Dirac equation by changing the representation of the matrix. For the case of the free particle, however, we have no prescription to distinguish one representation from the others. Or rather, we must consider that the true formalism for the free particle is quite independent of the representation employed.

Now, take the two different representations, and write down the Dirac equations in each representations respectively,

$$(\Gamma_{1\mu}\partial_\mu+\kappa)\varphi_1=0,$$

$$\Gamma_{2\mu}=e^{-2i\beta\gamma_5}\Gamma_{1\mu},$$

$$(\Gamma_{2\mu}\partial_\mu+\kappa)\varphi_2=0,$$

where β is an arbitrary C-number constant. The wave functions φ_1 and φ_2 are connected by the relation

$$\varphi_2=e^{i\beta\gamma_5}\varphi_1=\varphi_1 cos\beta+(i\gamma_5\varphi_1)sin\beta. \quad \cdots\cdots\cdots (24)$$

As we can interpret that the wave function φ_1 and $i\gamma_5\varphi_1$ have the opposite parity, to change the representation of the matrix γ is equivalent to add the opposite parity part to the wave function. Therefore, employing the suitable representation, we can consider the pure parity wave function. Let such a matrix be γ_μ and the corresponding wave function be φ_0. The arbitrary wave function φ can be expressed by using the pure parity wave functions φ_0 and $i\gamma_5\varphi_0$ as follows;

$$\varphi=\varphi_0 cos\beta+(i\gamma_5\varphi_0)sin\beta.$$

The relation between φ_0 and $i\gamma_5\varphi_0$ corresponds to take the value of the constant β equal to $\pi/2$. Thus, this formalism can be so interpreted that any unit radius vectors in the two dimentional plane having the wave functions φ_0 and $i\gamma_5\varphi_0$ as the orthogonal unit vectors are to be acceptable with the same rights as the wave functions of the Fermions satisfying the Dirac equations.

Therefore, it seems natural to introduce the following assumptions:

ASSUMPTION I

Though the wave function of the Fermion φ has the form

$$\varphi=\varphi_0 cos\beta+(i\gamma_5\varphi_0)sin\beta,$$

when it interacts with the Bosons, the constituent parts φ_0 and $i\gamma_5\varphi_0$ interact separately. In other words, the constituent parts φ_0 and $i\gamma_5\varphi_0$ give the independent contributions to the interaction Lagrangian. In constructing the Lagrangian, φ_0 and $i\gamma_5\varphi_0$ should be

weighted by the factors $\cos\beta$ *and* $\sin\beta$ *respectively.*

ASSUMPTION II

The electromagnetic and the strong interactions should not depend on the value of the constant β.

These assumptions mean that, in our model, φ_0 and $i\gamma_5\varphi_0$ are no longer of parametric but correspond to the real existence.

§ 3. INTERACTION

As the form of the interaction between nucleon and pion is of the form

$$\bar{\psi}\tau\pi\psi,$$

even if we rewrite $\tau\pi$ briefly as π and regard π as if it is a wave function for the neutral particle, we can develope the whole theory without the loss of generality. We will treat pion and photon as the neutral spin 0 and 1 particles respectively.

Let us put

$$\varphi_0=\varphi_+,\ i\gamma_5\varphi_0=\varphi_- \qquad (25)$$

then, as we have no method to find out which of the wave functions φ_+ and φ_- corresponds to the actual nucleon field at all, we can suppose that

ASSUMPTION III:

The interaction Lagrangian must be formally symmetric with respect to the wave functions φ_+ *and* φ_-. *Of course, the weight factors* $\cos\beta$ *and* $\sin\beta$ *must be multiplied to* φ_+ *and* φ_- *respectively.*

Under this assumption, the reasonable Lagrangian density can be written as

$$L=g_1[\cos^2\beta\bar{\varphi}_+\Gamma\varphi_+B+\sin^2\beta\bar{\varphi}_-\Gamma\varphi_-B]$$
$$+g_2\cos\beta\sin\beta[\bar{\varphi}_+\Gamma\varphi_-B+\bar{\varphi}_-\Gamma\varphi_+B], \qquad (26)$$

where B denotes the wave function of the Boson field. It is the direct result of the assumption III that we must take the same coupling constants to the terms $\bar{\varphi}_+\Gamma\varphi_+B$ and $\bar{\varphi}_-\Gamma\varphi_-B$ and to the terms $\bar{\varphi}_+\Gamma\varphi_-B$ and $\bar{\varphi}_-\Gamma\varphi_+B$.
In order to make the Lagrangian hermitic, g_1 and g_2 must be real. Γ are the matrices containing γ. Their explicit forms are as follows:

B	Γ	
spin 0, direct	1 or $i\gamma_5$	(27, 1)
spin 0, derivative	$i\gamma_\mu$ or $i\gamma_5\gamma_\mu$	(27, 2)
spin 1, direct	$i\gamma_\mu$ or $i\gamma_5\gamma_\mu$	
spin 1, derivative	$\sigma_{\mu\nu}$	(27, 3)

Using the relations (25), the Lagrangian density (26) can be written as

$$L'=g_1\bar{\varphi}_0\gamma_5(\cos^2\beta\gamma_5\Gamma-\sin^2\beta\Gamma\gamma_5)\varphi_0B$$
$$+ig_2\cos\beta\sin\beta\bar{\varphi}_0(\Gamma\gamma_5+\gamma_5\Gamma)\varphi_0B. \quad\cdots\cdots\cdots (26')$$

When Γ has the form (27, 1) or (27, 3), as

$$[\Gamma,\ \gamma_5]_-=0$$

we have

$$L'=g_1\cos2\beta\bar{\varphi}_0\Gamma\varphi_0B+ig_2\sin2\beta\bar{\varphi}_0\gamma_5\Gamma\varphi_0B. \quad\cdots\cdots\cdots (28)$$

When Γ has the form (27, 2), as

$$\{\Gamma,\ \gamma_5\}_+=0,$$

we have

$$L'=g_1\bar{\varphi}_0\Gamma\varphi_0B. \quad\cdots\cdots\cdots (29)$$

As for the free particle, we can not distinguish φ_+ and φ_- and so the interaction (28) leads to the non-conservation of the parity. Besides, this interaction depends on the value of the parameter β. Due to the assumption I, this interaction can not become a strong interaction. On the other hand, the interaction (29) does conserve parity and does not depend on the value of the constant β. This interaction can be strong. In other words, if some interaction is found to be strong for some reason, experimentally or theoretically, it can not be of a scalar or tensor type. Thus, we can get rid of the scalar and tensor interactions from the strong interactions. As the electromagnetic and the pion-nucleon interactions are confirmed to be strong, the Pauli-term in the former and the direct interaction in the latter must be excluded.

If the hypothesis of the global symmetry is allowed for the strong interactions containing pions, the above conclusions can be applied to all of the strong interactions containing pions. Further, even when the hypothesis of the global symmetry does not hold, if we assume that the hyperons with the same strangeness have the same value of the parameter, the above conclusions can be applied to those interactions, too. It is easily proved that the requirement of the gauge invariance is assured for the electromagnetic interactions.

§4. DISCUSSIONS AND CONCLUSIONS

When we generalize the Dirac equation as was done in §2, we define the reducing operator R as

$$R=e^{[illegible]},$$

by requiring that the transformation must be unitary. But, mathematically, the requirement of the unitarity of the operator R is not always necessary, and the formalisms reduce to the ordinary one for the arbitrary value of the constant α. In this case,

however, due to the fact that the transformation is not unitary, the wave function obeying the ordinary Dirac equation is to be considered as a parametric one. In this case, therefore, it is meaningless to separate the wave function φ in § 2 into φ_0 and $i\gamma_5\varphi_0$. The reducing operator should be understood to give the method to simplify the calculation. In the interaction Lagrangian, the Fermion wave functions must always have the form

$$\varphi = e^{\alpha\gamma_5}\varphi_0,$$

where α is an arbitrary real C-number constant.

Of course, we can proceed the discussion for this case along the same line as was done in § 2. Firstly, we write

$$\varphi = \cosh\alpha(1+\gamma_5 \tanh\alpha)\varphi_0.$$

For $\alpha=0$, we have

$$\varphi = \varphi_0.$$

This means that we can consider φ_0 as the wave function corresponding to the actual particle. As for the other wave function, if we make $\alpha\rightarrow\infty$, we have

$$\varphi = (\lim_{\alpha\to\infty} \cosh\alpha)(1+\gamma_5)\varphi_0.$$

This case corresponds to introduce the indefinite factor into the wave function φ. In general, such an introduction is not so favourable. But, as this unfavourable factor vanishes at the end of the formalism, there might be no contradiction even though we suppose that the other wave function may be

$$\varphi' = (1+\gamma_5)\varphi_0.$$

Of course, there must be no method to distinguish φ_0 and φ' for the case of the free particle. This means that both of the wave functions φ_0 and $\varphi'=(1+\gamma_5)\varphi_0$ must satisfy the Dirac equations at the same time. It is easily proved that, in order that both of φ_0 and $(1+\gamma_5)\ \varphi_0$ must satitfy the Dirac equations at the same time, κ must vanish. This corresponds to the case of the neutrino. It is well known that such a formalism as a two-component theory of the neutrino is admittable only when the mass of the particle considered is zero.

When the reducing operator is unitary, we can obtain the selection rule for the direct and the derivative couplings by setting the assumptions introduced in § 3 and § 4. At the present stage of the theory, the existing selection rules are those between the different elementary particles and there is no selection rule between direct and derivative interactions. As the direct and the derivative interactions are so constructed as both of them have the same rights as the allowable interactions between the particles, the selection rules required must have the different form from the existing ones. The difference of direct and derivative coupling is apparently comes from the number of the γ matrices, the assumptions in § 2 and § 3 seem to have a good uniqueness.

ACKNOWLEDGMENT

The author wishes to express his heartful thanks to Prof. Ozaki for his kind helps and discussions.

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ON LINEAR TOPOLOGICAL SPACES

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(Received Nov. 29, 1961)

This is a collection of results on linear topological spaces, particularly on tonnelé or bornologic spaces. The most part of the work was done during 1956–1959, at the University of Tokyo. The author wishes to express his hearty thanks to Professors K. Yosida and S. Irie for their kind advices.

1. Embedding in tonnelé or bornologic spaces.

N. Bourbaki introduced in [1] the notion of tonnelé spaces and remarked that an arbitrary locally convex complete space can be considered as a closed subspace of some tonnelé space. In this section we show more generally that an arbitrary locally convex (not necessarily complete) space is a closed subspace of some tonnelé space, and show an analogous property for bornologic spaces under some restrictions.

LEMMA 1.1 (*Mackey-Dieudonné*) *Every subspace F with co-dimension 1 of a tonnelé space E is also tonnelé.*

PROOF. Let U be an arbitrary closed convex circular absorbing set of F. When $\overline{U}$(=the closure of U in E) contains some point $x_0 \notin F$, $\overline{U}$ is a neighbourhood of zero in E, since $\overline{U}$ is a closed convex circular absorbing set of the tonnelé space E. In this case $\overline{U}=U\cap F$ is a neighbourhood of zero in F. When $\overline{U}\subset F$, we have $U=\overline{U}\cap F=\overline{U}$. In this case, for an element x_0 of E such that $x_0 \notin F$, the set $\widetilde{U}=\{\lambda x_0+y;\ |\lambda|\leq 1,\ y\in U\}$ is a closed convex circular absorbing set of E. Hence $\widetilde{U}$ is a neighbourhood of zero in E, and $U=\widetilde{U}\cap F$ is a neighbourhood of zero in F.

Thus in every case U is a neighbourhood of zero in F.

LEMMA 1.2 *An arbitrary complete locally convex space E is a closed subspace of some product space ΠE_α of Banach spaces E_α.*

This is a well known fact and we show outline of the proof. Pick up a fundamental semi-norm system $\{P_\alpha\}$. For each semi-norm P_α, we define a normed space $F_\alpha=E/N_\alpha$, with the norm induced by P_α, where $N_\alpha=\{x\in E;\ P_\alpha(x)=0\}$. Then the space E is embedded in natural way into the product space of the completions of F_α's.

THEOREM 1.1 *An arbitrary locally convex space E is a closed subspace of some tonnelé space.*

PROOF. Let F be a tonnelé space containing the completion of E, and $\{e_\alpha\}$ be a Hamel base (=maximal linearly independent set) of a compliment space F^C of F.

For any index α, let F_α be the subspace of F generated by E and $\{e_\beta\}_{\beta\neq\alpha}$. Since F_α is of co-dimension 1, it is tonnelé.

The space E is naturally embedded in ΠF_α, which is a tonnelé space. Precisely, $x \in E$ is identified to $(x_\alpha) \in \Pi F_\alpha$ if and only if $x_\alpha = x$ for any index α.

If some directed set $(x_\alpha)^\varphi = (x_\alpha^\varphi)$ in E converges to $(y_\alpha) \in \Pi F_\alpha$, then for any α, x_α^φ converges to y_α. Since x_α^φ is independent of α, y_α is independent of α. Thus $y_\alpha = y \in \bigcap_\alpha F_\alpha = E$, which means the closedness of E in ΠF_α. q. e. d.

Since the dual F' of a tonnelé space F is semi-reflexive with respect to the Mackey topology $\tau(F', F)$, we have the following corollary dual of the theorem.

COROLLARY. *An arbitrary locally convex space E is a quotient space of some semi-reflexive space.*

PROOF. Without losing the generality we may suppose that the topology of E is equal to $\sigma(E, E')$. By virtue of the theorem 1.1 there exists a tonnelé space F which contains E' with the topology $\sigma(E', E)$ as a closed subspace. Then the space $E \cong$ (*the dual of E' with respect to $\tau(E', E)$*)$\cong F'/E'^{\perp}$. q. e. d.

Concerning bornologic spaces, the same lemma as lemma 1.1 holds good. As to lemma 1.2, the product space ΠE_α of Banach spaces E_α is bornologic if the power of the index set is less than some cardinal number d. (See Köthe [6]). We denote by $\aleph$ the dimension of E (= the power of a maximal linearly independent set). Then, we have the relation (*the power of* $\{\alpha\}$)$\leq 2^{\aleph}$, since a semi-norm P_α is uniquely determined by a subset $\{x \in E;\ P_\alpha(x) < 1\}$ of E. Suppose that $\aleph < d$. By virtue of the theorem of Mackey-Ulam (see Köthe [6]), $\aleph < d$ implies $2^{\aleph} < d$. Therefore, we have the following lemma.

LEMMA 1.3. *An arbitrary complete locally convex space E with the dimension $\aleph < d$ is a closed subspace of some bornologic space F.*

For the bornologic space F, dim $F < \aleph^{\aleph}$, hence dim $F < d$. Therefore the space E is naturally embedded as a closed subspace in some product space ΠF_α of bornologic spaces F_α, in a same way as the proof of the theorem 1.1. The power of the index set $\{\alpha\}$ is $\leq$ dim F, hence the space ΠF_α is bornologic. Thus we have the following theorem.

THEOREM 1.2. *An arbitrary locally convex space with the dimension $< d$ is a closed subspace of some bornologic space.*

In particular, *an arbitrary locally convex separable space is a closed subspace of some bornologic space*, since the dimension of such a space is not larger than $2^{2^{\aleph_0}}$.

Dually we have a corollary.

COROLLARY. *A separable tonnelé space is a quotient space of some complete space.*

More precisely, a locally convex space E, with the topology $\beta(E, F)$ for some subspace F of E', is a quotient space of some complete space. The author cannot tell whether any locally convex space with Mackey topology is a quotient space of some complete space or not.

2. Tonnelé topology defined by locally convex topology.

Let E be a fixed linear space and τ be a locally convex topology on E. We consider the set $\{\tau_\alpha\}$ of all tonnelé topology which are stronger than τ. The limit inductive topology $\bar{\tau}=\bigcap\tau_\alpha$ of $\{\tau_\alpha\}$, that is, the strongest locally convex topology on E weaker than every τ_α, is uniquely defined. The topology $\bar{\tau}$ is the weakest among the tonnelé topologies stronger than τ.

DEFINITION 2.1. For a locally convex topology τ (on E), the weakest tonnelé topology $\bar{\tau}$ stronger than τ is called *the tonnelé topology defined by* τ.

We denote by $E(\tau)$ (or E_τ) the linear space E with the topology τ. Then the strong topology $\beta(E, E(\tau)')$ is the topology generated by a fundamental neighbourhood system of zero which are all τ-closed convex absorbing sets. We denote by τ^1 the strong topology $\beta(E, E(\tau)')$.

For any ordinal number α we define $\tau^\alpha=\tau^{\beta,1}(=\beta(E,E(\tau^\beta)')$if $\alpha=\beta+1$, and $=$ the limit projective topology $\bigcup\limits_{\beta<\alpha}\tau^\beta$ if α is a limit number.

For any τ there exists an ordinal number α such that $\tau^\alpha=\tau^{\alpha+1}$. Evidently such a topology τ^α is identical with the tonnelé topology $\bar{\tau}$ defined by τ.

THEOREM 2.1. *Let E be a linear space with a locally convex topology τ. Then the τ-completion of E contains the $\bar{\tau}$-completion of it, where $\bar{\tau}$ is the tonnelé topology defined by τ.*

PROOF. We shall prove using transfinite induction. We assume that the τ^γ-completion of E contains the τ^β-completion of it for any γ and any β such that $\gamma<\beta<\alpha$. When $\alpha=\beta+1$ for some β, the τ^α-completion of E is contained in the τ^β-completion of it, by virtue of Grothendieck's theorem. When α is a limit number, then $E(\tau^\alpha)'=\bigcup\limits_{\beta<\alpha}E(\tau^\beta)'$ and any equi-continuous set in $E(\tau^\alpha)'$ is contained and equi-continuous in some $E(\tau^\beta)'$.

On the other hand, the τ^γ-completion of E (= the set of all linear functionals weakly continuous on each τ^γ-equi-continuous set) contains the τ^β-completion of E (= the set of all linear functionals weakly continuous on each τ^β-equi-continuous set), by the assumptions of induction. Hence a linear functional which is weakly continuous on each equi-continuous set in some $E(\tau^\gamma)'$ is, if extensible, uniquely extended to a linear functional which is weakly continuous on each equi-continuous set of $E(\tau^\beta)'$ for $\gamma<\beta<\alpha$.

Let l be an arbitrary element of the τ^α-completion of E. Then l is weakly continuous on each τ^α-equi-continuous set in $E(\tau^\alpha)'$, hence it is weakly continuous on each τ^γ-equi-continuous set in $E(\tau^\gamma)'$ for any $\gamma<\alpha$. Since the restriction l^γ of l to $E(\tau^\gamma)'$ is uniquely extended to the restriction l^β of l to $E(\tau^\beta)'$ for $\gamma<\beta<\alpha$, we may identify $l^\gamma=l^\beta$. The relation $E(\tau^\alpha)'=\bigcup\limits_{\beta<\alpha}E(\tau^\beta)'$ implies that the inverse of the mapping: $l\rightarrow l^\beta$ is unique, hence the canonical mapping from the τ^α-completion of E to the τ^β-comletion of E is one-to-one. q. e. d.

Slightly modifying, we can verify the following corollary, a generalization of Grothendieck's theorem.

COROLLARY. *Let σ and τ be two locally convex topologies on some linear space E. The σ-completion of E contains the τ-completion of E if the following conditions are satisfied:*
1) $\sigma < \tau$,
2) a linear space F, such that $E'_\sigma \subseteq F \subset E'_\tau$ and the intersection of F with each weakly closed equi-continuous set in E'_τ is weakly closed, is necessarily identical with E'_τ.

3. Closed graph theorem and minimal topology.

In this note we discuss the closed graph theorem in abstract form. The result is essentially contained in [2], [4], [5] and [8].

LEMMA 3.1. *Let E and F be locally convex spaces, and u be a linear operator from E to F. (The domain of u is a whole space E.) u is a closed operator if and only if u is continuous from E to $F_\tau = F$ with some weaker separated topology τ.*

PROOF. Suppose that such a topology τ exists. If a directed set $\{x_\varphi\} \subset E$ converges to x in E, then $u(x_\varphi)$ converges to y in F_τ. If moreover $u(x_\varphi)$ converges to y' in F, y coinsides with y' since the canonical mapping $F \to F_\tau$ is continuous. This means that u is a closed operator. q. e. d.

Conversely, we suppose that u is not continuous with respect to any weaker separated topology. Then the topology generated by a system of neighbourhoods $\{U + u(V): U = \text{neighbourhood of } F, V = \text{neighbourhood of } E\}$ is not separated. Hence there exists some $x_0 \in F, x_0 \neq 0$, such that $x_0 \in U + u(V)$ for any neighbourhood U of zero in E and any neighbourhood V of zero in F. Therefore there exists a directed set $\{x_{U,V}\}$ such that $x_{U,V} \in V$ and $u(x_{U,V}) \in U + x_0$, that is, $x_{U,V} \to 0$, $u(x_{U,V}) \to x_0 \neq 0$. This means that u is not a closed operator. q. e. d.

We consider some property (α) of locally convex spaces. Property (α) is called *invariant under finite (limit) inductive operations* if the following condition is satisfied: (A) *Let $\{E_i\}$ be a set of finite (infinite) (α)-spaces, and u_i be a linear operator from E_i to same linear space E for any i, and some u_j be an operator onto E. Then the space E, given the strongest locally convex topology such that each u_i is continuous, is also an (α)-space.*

For example, the property of "*tonnelé*" is invariant under limit inductive operations.

PROPOSITION 3.1. *Let (α) be invariant under finite inductive operations. Any closed linear operator from any (α)-space to a fixed (α)-space E is continuous if any only if there is no separated (α)-topology on E weaker than the original topology.*

The proof follows immediately from the lemma 3.1.

COROLLARY. *Let E be a $\sigma(E, E')$-complete space, that is, E is isomorphic to a direct product space of finite dimensional spaces. Then any closed linear operator from a locally convex space to E is continuous.*

PROOF. On such a space E, there exists no weaker separated topology. Hence if we consider the property (α) as the property of locally convex topologies, our corollary is obtained immediately. q. e. d.

We call an (α)-topology on E the *(α)-minimal topology* if there is no separated

weaker (α)-topology. Then we may say that E *has the property of the closed graph theorem with respect to* (α) *if and only if* E *is endowed with an* (α)*-minimal topology.*

A space E which is a limit inductive of (α)-spaces $\{E_\lambda\}$ is called *an* $(\bar{\alpha})$*-space.* Thus we have an extension $(\bar{\alpha})$ of the class of (α)-spaces. Evidently property $(\bar{\alpha})$ is invariant under limit inductive operations.

For any locally convex topology τ on E, there exists the unique $(\bar{\alpha})$-topology $\bar{\tau}$ which is the weakest among the $(\bar{\alpha})$-topologies stronger than τ. In fact, $\bar{\tau}$ is the limit inductive topology of all $(\bar{\alpha})$-topologies stronger than τ.

We call $\bar{\tau}$ *the* $(\bar{\alpha})$*-topology defined by* τ.

THEOREM 3.1. *Let* E *be a locally convex space with the topology* τ. *Any linear closed operator* u *from any* $(\bar{\alpha})$*-space* F *to* E *is necessarily continuous if and only if the* $(\bar{\alpha})$*-topology* $\bar{\tau}$ *defined by* τ *is identical with the* $(\bar{\alpha})$*-topology* $\bar{\sigma}$ *defined by any separated locally convex topology* σ *on* E *weaker than* τ.

PROOF. Let u be a closed operator from an $(\bar{\alpha})$-space F to E. Then by virtue of lemma 3.1 there exists some separated locally convex topology σ weaker than τ, and u is continuous from F to $E(\sigma)$. Since $(\bar{\alpha})$ is invariant under finite inductive operations, u is continuous from F to $E(\bar{\sigma})$. Therefore u is continuous from F to $E(\tau)$ if $\bar{\sigma}>\tau$. Since $\sigma<\tau$, the condition $\bar{\sigma}>\tau$ means $\bar{\sigma}=\bar{\tau}$. Conversely, if there exists some separated locally convex topology σ weaker than τ such that $\bar{\sigma}\neq\bar{\tau}$, then the identity mapping u: $E(\bar{\sigma})\rightarrow E(\tau)$ is not continuous. Since u is a closed operator from $E(\sigma)$ to $E(\tau)$, it is also a closed operator from $E(\bar{\sigma})$ to $E(\tau)$. q. e. d.

We explain the case of tonnelé spaces. The property of tonnelé is invariant under limit inductive operations, that is, a limit inductive space of tonnelé spaces is also tonnelé. Therefore the extension of the class of tonnelé spaces by limit inductive operations is also the class of tonnelé spaces.

By virtue of the proposition 3.1, a tonnelé space E has a tonnelé-minimal topology τ if and only if any closed operator from any tonnelé space to E is continuous. Evidently a locally convex topology τ is tonnele-minimal if and only if the tonnelé topology $\bar{\sigma}$, defined by any locally convex topology σ weaker than τ, is identical with τ. In other words, E is a tonnelé-minimal space if and only if the following condition is satisfied: "for a $\sigma(E',E)$-dense subspace F of E', if the intersection of F with each $\sigma(E',E)$-closed equi-continuous set in E' is necessarily $\sigma(E',E)$-closed, then we have $F=E'$". Since a subspace with co-dimension 1 is dense or closed, for a tonnelé-minimal space E a subspace F of E' with co-dimension 1 is $\sigma(E',E)$-closed when the intersection of F with each $\sigma(E',E)$-closed equi-continuous set is closed. Therefore such a space E is complete.

In addition, we give the following proposition.

PROPOSITION 3.2. *Let* E *be a locally convex space and* E_0 *be its dense subspace. If a closed linear operator from* E *to a locally convex complete space* F *is continuous on* E_0, *then it is continuous on* E.

The proof is almost obvious. It is to be noted that *for a non-complete space* F *the*

above proposition is not true in general. In fact, let H be a dense subspace of E with co-dimension 1. (Such a subspace exists if and only if there exists some discontinuous linear functional on E. For instance, an infinite dimensional normed space has always such a subspace.) If F is the direct sum $H \oplus R^1$ of H and 1-dimensional space R^1, then the one-to-one linear operator u: $F \to E$, which is the identity map on H, is continuous. Therefore the inverse operator u^{-1}: $E \to F$ is a closed operator which is continuous on a dense subspace H, but it is not continuous.

4. A class of boundedly closed system.

A. Grothendieck showed in [3], lemma 9, that the following three conditions for a locally convex space E are equivalent to each other.

a) *All continuous linear operator from E to any Banach space is weakly compact.*

b) *For any convex circular neighbourhood V of the origin in E, there exists a convex circular neighbourhood $U \subset V$ such that the canonical mapping from the Banach space $\widehat{E}_U$ $=$the completion of the normed space E/N_U, $N_U=\{x \in E;\ \lambda x \in U$, for any scalar $\lambda\}$, with the unit ball U) to the Banach space $\widehat{E}_V$, is weakly compact.*

c) *For any convex circular equi-continuous weakly closed subset A of E', there exists a convex circular equi-continuous weakly closed subset B of E' such that A is a weakly compact set in the Banach space E'_B $(=$the normed space generated by B with unit ball $B)$.*

Spaces E satisfying the above property are a generalization of Schwartz spaces. In fact, if we replace the term "weakly compact" by "compact", then we have the definition of Schwartz spaces.

We shall prove that *the above conditions are equivalent to the following condition.*

d) *E is naturally embedded in a direct product ΠE_α of Banach spaces E_α (see § 1) such that each E_α is the dual space F'_α of some Banach space F_α and $E' = \Sigma F_\alpha / E^\perp$.*

PROOF. Let $\{V_\alpha\}$ be a fundamental system of neighbourhoods of the origin such that each V_α is convex circular and closed. Assume the condition b). Then there exists another fundamental system $\{U_\alpha\}$ of neighbourhoods such that for any α, $U_\alpha \subset V_\alpha$ and the canonical image of U_α in $\widehat{E}_{V_\alpha}$ is relatively weakly compact. Then the closure $\overline{U}_\alpha$ of the image of U_α is $\sigma(\widehat{E}_{V_\alpha}, \widehat{E}'_{V_\alpha})$-compact. Hence $(\widehat{E}_{V_\alpha})_{\overline{U}_\alpha}=$ the dual of the normed space $(E_{V_\alpha})'_{U_\alpha^0}$ $(=$ the space generated by U^0_α with the unit ball U^0_α, where $U^0_\alpha = \{x' \in (E_{V_\alpha})';\ \sup_{x \in U_\alpha} |\langle x, x' \rangle| \leq 1\})$.

We put $E_\alpha = (\widehat{E}_{V_\alpha})_{\overline{U}_\alpha}$ and $F_\alpha=$ the completion of $(E_{V_\alpha})'_{U_\alpha^0}$. For any two indices α and β such that $U_\alpha \subset U_\beta$ and $V_\alpha \subset V_\beta$, there exists the canonical mapping $u_{\beta\alpha}$: $F_\beta \to F_\alpha$.

For any F_α, by virtue of the condition c) there exists an F_β such that the mapping $u_{\beta\alpha}$ is weakly compact. Then the bi-transposed operator $''u_{\beta\alpha}$: $E'_\beta = F''_\beta \to F''_\alpha = E'_\alpha$, maps $E'_\beta = F''_\beta$ to F_α. For any pair x, y such that $''u_{\beta\alpha}(x) = y$, the element $x - y \in \Sigma E'_\alpha$ is contained in $E^\perp$, and therefore, we conclude that $\Sigma F_\alpha / E^\perp = \Sigma F''_\alpha / E^\perp$.

The converse part of our assertion is almost obvious. q. e. d.

From the condition d), if such a space E is complete, we have $E=(\Sigma F_\alpha/E^\perp)'$. In this case, its dual space E' with the Mackey topology $\tau(E',E)$ is a (β)-space ($=$ a limit inductive of Banach spaces), and E is the set of all linear functionals bounded on each equi-continuous set in E'. In particular, we have the following corollary.

COROLLARY. *A Schwartz space E is complete if and only if it is the set of all linear functionals bounded on each equi-continuous set in E'.*

5. Closedness and quasi-closedness.

Let E be a locally convex space and $\mathfrak{B}$ be a system of bounded sets. A subset A of E is said to be *quasi-closed* (or *quasi-complete*) *with respect to* $\mathfrak{B}$ if, for any $B\in\mathfrak{B}$, $A\cap B$ is closed (or complete) in B. When $\mathfrak{B}$ is the set of all closed bounded sets, we shall omit the term "with respect to $\mathfrak{B}$" for the sake of brevity. For instance, when $\mathfrak{B}$ is the set of all metrizable precompact sets, a subset A is quasi-closed with respect to $\mathfrak{B}$ if and only if A is sequentially closed.

The problem, to ask in what case the closedness (or completeness) follows from the quasi-closedness (or quasi-completeness), may be one of the most important but difficult problems in the theory of linear topological spaces. (For instance, this is closely connected with the closed graph theorems on tonnelé spaces. See § 2). It is well known that in case of (DF)-spaces, the completeness follows from the quasi-completeness, and in case of (F)-spaces, the closedness of subspaces follows from the quasi-closedness. We shall give generalizations of these cases.

PROPOSITION 5.1. *A locally convex space E is closed in the bidual E'' with respect to the natural topology ($=$ topology of uniform convergence on each equi-continuous set in E') if and only if E is quasi-complete.*

PROOF. Let $\bar{E}$ be the closure of E in E''. For any $x\in E''$, there exists a convex circular bounded set $B\subset E$ such that $x\in\tilde{B}=\sigma(E'',E')$-closure of B. If $x\in\bar{E}$, then $x\in\tilde{B}\cap\bar{E}=\sigma(\bar{E},E')$-closure of B. Since the dual of $\bar{E}$ with respect to the natural topology is equal to E', $\tilde{B}\cap\bar{E}=$ the closure of B with respect to the natural topology, in which x is contained. The quasi-completeness of E implies $\tilde{B}\cap\bar{E}\subset E$, and therefore $\bar{E}\subset E$.

Conversely, if E is closed in E'' with respect to the natural topology, then E is quasi-complete since E'' is quasi-complete with respect to $\mathfrak{B}=\{\tilde{B}=\sigma(E'',E')$-closure of B; B is any bounded set in $E\}$. q. e. d.

By virtue of the above proposition, for a space E such that E'' is complete with respect to the natural topology, the completeness follows from the quasi-completeness. This is a usual method to prove the completeness of quasi-complete (DF)-spaces. (See Köthe [6].)

We say that *a sequence $\{x_k\}$ converges to x_∞ in the sense of Mackey* if, for some infinitely increasing sequence $\{\lambda_k\}$ of scalars, the set $\{\lambda_k(x_\infty-x_k)\}$ is bounded.

PROPOSITION 5.2. (Mackey) *Let E be a bornologic space and H be its subspace with co-dimension* 1. *If H is sequentially closed in the sense of Mackey, then it is closed.*

PROOF. Assume that H is not closed. We pick up an element x_0 of $E, \notin H$. Let B be an arbitrary bounded set of E. Then for some positive number λ_0 and bounded set $A \subset H$, the set B is contained in the set $\{x=y+\lambda x_0;\ y\in A,\ |\lambda|\leq\lambda_0\}$. In fact, if, for $x_k=y_k+\lambda_k x_0 \in B$, $|\lambda_k|$ is infinitely increasing, then $-y_k/\lambda_k=x_0-x_k/\lambda_k\rightarrow x_0$ in the sense of Mackey, which contradicts to the sequential closedness of H. Hence there exists a scalar λ_0 and a subset A of H such that $B\subset\{x=y+\lambda x_0;\ y\in A,\ |\lambda|\leq\lambda_0\}$. We see easily that A can be chosen to be bounded.

From the above fact, the convex circular set $V=\{x=y+\lambda x_0;\ y\in H,\ |\lambda|\leq 1\}$ absorbs each bounded set in E. In other words, there exists some bounded linear functional u on E which is identically zero on H and $u(x_0)\neq 0$. Since we assume that H is not closed, that is, H is dense in E, E is not bornologic. This is a contradiction. q. e. d.

Let E be a locally convex space and H be its subspace which is sequentially closed in the sense of Mackey. If for an element $x_0\in E,\ \notin H$, the space $H\dot{+}x_0$ generated by H and x_0 is a bornologic subspace of E, then H is closed in $H\dot{+}x_0$ by virtue of the above proposition, that is, the closure of H does not contain x_0. Hence, if $H\dot{+}x$ is bornologic for any $x\in E,\ \notin H$, then H is closed in E. Thus we have the following corollary.

COROLLARY. *Let E be a locally convex space whose any subspace is bornologic. Then an arbitrary sequentially closed subspace in the sense of Mackey is always closed.*

This gives another (and more complicated) proof of the fact that an arbitrary sequentially closed subspace of a locally convex metrizable space is closed.

It is well known that a locally convex space which possesses a dense tonnelé subspace is tonnelé too. This is not true for bornologic case. In fact, we know many example of bornologic and sequentially complete, but non-complete spaces. Such spaces are densely contained in (tonnelé and) non-bornologic spaces. In the following, we shall show *the existence of tonnelé (DF) and non-bornologic spaces*, which gives the negative answer to the problem 3) in [3].

G. Köthe gave an example of a non-complete limit inductive space E of countable Banach spaces $\{E_n\}$ (See [6]). Let x_0 be an element of the completion $\hat{E}$ of E, such that $x_0\notin E$. Since E'', which is the dual space of an (F)-space, possesses a fundamental sequence $\{B_n\}$ of bounded sets (that is, an arbitrary bounded set $B\subset E''$ is contained in some (B_n), the subspace $\tilde{E}=E\dot{+}x_0$ of E generated by E and x_0 possesses a fundamental sequence $\{B_n\cap\tilde{E}\}$ of bounded sets.

Since $\tilde{E}$ contains a dense tonnelé subspace $E, \tilde{E}$ itself is tonnelé. Therefore $\tilde{E}$ is a (DF)-space. If E is sequentially complete in the sense of Mackey, $\tilde{E}$ is not bornologic by virtue of the proposition 5.2. Hence it suffices to prove the following lemma.

LEMMA 5.1. *The limit inductive space E of a sequence $\{E_n\}$ of Banach spaces is sequentially complete in the sense of Mackey.*

PROOF. From the assumption, we have $E=(\Sigma E_n)/N$, where N is a closed subspace of ΣE_n. Hence $E'=N^{\perp}\subset \Pi E'_n$ and $E''=(\Sigma E''_n)/N^{\perp\perp}$, since the topology of E' is the topology induced by $\Pi E'_n$. Since E'' is complete, the completion $\hat{E}$ of E is contained in E''. If E is complete, our assertion is obvious. We consider the case $\hat{E}\neq E$. Let x_0 be an element of $\hat{E}$ such that $x_0 \notin E$. If a sequence $\{x_k\}\subset E$ converges to x_0 in the sense of Mackey, for some infinitely increasing sequence λ_k and a bounded set $B\subset \hat{E}$, we have $\{\lambda_k(x_0-x_k)\}\subset B$. Since B and $\{x_k\}$ are bounded sets of E'', we may assume, without losing the generality, that B and $\{x_k\}$ are contained in the unit ball of some E''_n, that is, $\|x_0-x_k\|_n \leq 1/\lambda_k$, where $\|\ \|_n$ denotes the norm of E''_n. Therefore x_k converges to x_0 in E_n. Since E_n is complete, we have $x_0\in E_n$, which contradicts to the assumption that $x_0 \notin E$. q. e. d.

We shall give *an example of semi-reflexive spaces which are not complete with respect to the Mackey topology.* (See [6] p. 311.) A locally convex space is semi-reflexive if and only if it is quasi-complete with respect to the weak topology (in other words, its any convex closed bounded set is weakly compact.)

EXAMPLE. $\aleph_0$, $\aleph_1$, or $\aleph_2$ denotes the first, second or third infinite cardinal number respectively. Let A be a set of power $\aleph_2$. We denote by $\omega(A)$ the direct product $\prod_{\alpha\in A} R_\alpha$ of each one dimensional space R_α, and denote by $\omega_0(A)$ the set $\{(x_\alpha)\in\omega(A);\ x_\alpha=0$ except countable indices $\alpha\}$. Then $\omega_0(A)$ is a tonnelé subspace of $\omega(A)$.

Let $E=\omega_0(A)\dot{+}1_A$ (that is, the space generated by $\omega_0(A)$ and constant function $1_A=$ the elements of $\omega(A)$ whose coordinates are identically 1). Then the space E, with the topology induced by $\omega(A)$, is tonnelé, and therefore the dual $E'(=\sum_{\alpha\in A} R_\alpha)$ is semi-reflexive with respect to the Mackey topology $\tau(E',E)$. We shall prove that E' is not complete with respect to the Mackey topology $\tau(E',E)$ We need the following lemma.

LEMMA 5.2. *Let B be a bounded set in $\omega_0(A)$. If the $\sigma(\omega(A), \sum_{\alpha\in A} R_\alpha)$-closure $\bar{B}$ of B contains 1_A, then it necessarily contains another element of $\omega(A)$ which is not contained in $\omega_0(A)$.*

By virtue of the above lemma, for any convex circular $\sigma(E',E)$-compact set B of E, the intersection $B\cap\omega_0(A)$ is closed in E. Hence the linear functional u on E, such that $u(1_A)=1$ and $u(x)=0$ for any $x\in\omega_0(A)$, is $\sigma(E',E)$-continuous on each convex circular $\sigma(E',E)$ compact set in E, that is, u is contained in the $\tau(E',E)$ completion of E'. However, u is not contained in E' since $\omega_0(A)$ is $\sigma(E',E)$-dense in E.

PROOF OF THE LEMMA. Let A_1 be a subset of A with the power $\aleph_1$. Then for any $\alpha\in A_1$, there exists a countable subset $B_1^\alpha=\{(x_\beta)_n(=(x_\beta^{(n)}));\ n=1,2,\cdots\}$ of B such that $|x_\alpha^{(n)}-1|<1/n$. We put $B_1=\bigcap_{\alpha\in A_1} B_1^\alpha$ and $A_2=\{\beta\in A;$ the β-th coordinate x_β of some element $(x_\beta^{(n)})$ of B_1 is not zero$\}$. Then A_2 is of power $\aleph_1$, since the coordinates of an element $(x_\beta^{(n)})$ of $\omega_0(A)$ are zero except countable indices.

In the similar way as above we can construct the sequence B_n of subsets of B

and the sequence A_n of sets of indices with power $\aleph_1$, satisfying

1) for any $\beta \in A_n$ and any $\varepsilon > 0$, there exists some $(x_\alpha) \in B_n$ such that $|x_\beta - 1| < \varepsilon$,

2) for any index $\beta \notin A_{n+1}$, the β-th coordinate x_β of any element (x_α) of B_n is zero.

Therefore for the sets $A_\infty = \bigcup_{n=1}^{\infty} A_n$ and $B_\infty = \bigcup_{n=1}^{\infty} B_n \subset B$, we have

1)′ for any $\beta \in A_\infty$ and any $\varepsilon > 0$, there exists some $(x_\alpha) \in B_\infty$ such that $|x_\beta - 1| < \varepsilon$,

2)′ for any index $\beta \notin A_\infty$, the β-th coordinate x_β of any element (x_α) of B_∞ is zero.

The conditions 1)′ and 2)′ imply that the $\sigma(\omega(A), \sum_{\alpha \in A} R_\alpha)$-closure of B_∞ contains the element 1_{A_∞} of $\omega(A)$, whose α-th coordinate is equal to 1 for $\alpha \in A_\infty$, is equal to 0 for $\alpha \notin A_\infty$. Since the power of A_∞ is $\aleph_1$, the element 1_{A_∞} is a required element.

q. e. d.

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THE INFERENCE THEORY IN MULTIVARIATE RANDOM EFFECT MODEL (I)

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(Received December 1, 1961)

1. Introduction.

In this paper we shall be concerned with the estimation of the parameters in the multivariate random effect model. The multivariate random effect model is understood to be a model where the observations are given by the multi-dimensional vectors and consequently the treatment effects, which are the normal variables, are also the multi-dimensional vectors. We shall discuss this problem under certain restrictions for the covariance matrices, while it seems to be more desirable to prove the problem under more general assumptions free from these restrictions.

Concerning the theory of estimation in the similar model as ours, the author should like to mention here in the first place the work of S. N. Roy and R. Gnanadesikan [5][1], in which their detailed discussions are concerned with the restricted model having the treatment effects whose covariance matrices are proportional to each other, and secondly the work of F. Grybill and R. A. Hultquist [9] which is concerned with the case of the univariate model.

The main results of this paper are Theorem 4.1 and Theorem 6.2. The former theorem gives the necessary and sufficient condition for the covariance matrices to be estimable, under our resticted model, while the latter gives the theorem concerning the completeness of the family of the distributions of the sufficient statistics in our concern. Section 3 is devoted to the derivation of the covariance matrix of all observations, and Section 5 to the discussion of some properties concerning the characteristic roots of covariance matrix, which seems to be crucial for the estimation theory under our model.

2. Preliminaries.

Let $\mathbf{Y}(N\times p)$ be a set of N observable stochastic p-dimensional vectors whose model equation is given by the following.

$$\mathbf{Y}(N\times p)=\mathbf{X}_0\mathbf{B}_0(1\times p)+\sum_{i=1}^{k}\mathbf{X}_i(N\times m_i)\mathbf{B}_i(m_i\times p)+\mathbf{X}_{k+1}(N\times N)\mathbf{B}_{k+1}(N\times p), \tag{2.1}$$

where we assume

(i) $\mathbf{B}_0(1\times p)=[\mu_1, \mu_2, \cdots, \mu_p]$ is a p-dimensional vector with the fixed but unknown constants μ_i's$(i=1, 2, \cdots, p)$;

(ii) $\mathbf{B}_i(m_i\times p)$ is a random sample of size m_i from the p-variate normal population $\mathrm{N}[\mathbf{O}(1\times p), \Sigma_i(p\times p)]$ for $i=1, 2, \cdots, k+1$;

1) Numbers in brackets refer to the references of the end of the paper.

(iii) $\mathbf{B}_{k+1}(N\times p)$, which denotes the error term, is a random sample of size N from the p-variate normal population $\mathrm{N}[\mathbf{O}(1\times p), \Sigma_{k+1}(p\times p)]$;

(iv) $\mathbf{B}_i$'s $(i=1, 2, \cdots, k)$ and $\mathbf{B}_{k+1}(N\times p)$ are mutually independent;

(v) $\mathbf{X}_0(N\times 1)=\mathbf{1}(N\times 1)$ is a N-dimensional vector of 1's, $\mathbf{X}_i$'s $(i=1, 2, \cdots, k)$ are the matrices of known constants, and $\mathbf{X}_{k+1}(N\times N)=\mathbf{I}(N\times N)$ is the identity matrix.

In what follows for the sake of simplicity we write sometimes $\mathbf{B}_0, \mathbf{B}_i$, $\mathbf{B}_{k+1}$ and X_i etc. instead of $\mathbf{B}_0(1\times p), \mathbf{B}_i(m_i\times p)$, $\mathbf{B}_{k+1}(N\times p)$ and $X_i(N\times m_i)$ etc..

Throughout this paper we shall write $n\times n$ identity matrix as $\mathbf{I}(n\times n)$, $\mathbf{E}(n\times n)$ denotes the $n\times n$ matrix with the elements all equal to 1.

Let $\mathbf{H}(n\times n)$ be the $n\times n$ matrix with the elements all equal to zero except for the (1, 1)-element equal to 1. $\mathbf{A}_i(N\times N)$ denotes $\mathbf{X}_i\mathbf{X}_i'$ for $i=1, 2, \cdots, k+1$ and $\mathbf{A}_{k+1}$ is equal to $\mathbf{I}(N\times N)$.

Further let $\mathbf{P}(N\times N)$ be defined as any orthogonal matrix whose elements in the first row are all equal to $\frac{1}{\sqrt{N}}$, and also let $\mathbf{Q}(p\times p)$ be any orthogonal matrix whose elements in the first row are all equal to $\frac{1}{\sqrt{p}}$.

Then we have easily

$$\mathbf{P}(N\times N)\mathbf{E}(N\times N)\mathbf{P}'(N\times N)=N\mathbf{H}(N\times N) \tag{2. 2}$$

and

$$\mathbf{Q}(p\times p)\mathbf{E}(p\times p)\mathbf{Q}'(p\times p)=p\mathbf{H}(p\times p). \tag{2. 3}$$

In this paper, the Kronecker product of two matrices are defined in the way reverse to the usual ones for the sake of convenience. Thus for $\mathbf{C}=(C_{ij})$ and $\mathbf{D}=(d_{ij})$, the Kronecker product denoted by $\mathbf{C}\otimes\mathbf{D}$ is defined as the matrix $(\mathbf{C}d_{ij})$. The Kronecker product of any number of matrices is defined as the natural generalization of two matrices. And we shall make use of the well-known relations concerning the Kronecker product of two matrices such as $(\mathbf{C}\otimes\mathbf{D})(\mathbf{L}\otimes\mathbf{M})=\mathbf{CL}\otimes\mathbf{DM}$, $(\mathbf{C}\otimes\mathbf{D})^{-1}=\mathbf{C}^{-1}\otimes\mathbf{D}^{-1}$, $(\mathbf{C}\otimes\mathbf{D})'=\mathbf{C}'\otimes\mathbf{D}'$, and their generalization to the products of any number of matrices without mentioning explicitly.

3. Covariance matrix.

At first we observe

THEOREM 3. 1. *Let*

$$\mathbf{Y}(N\times p)=[\mathrm{Y}_1, \mathrm{Y}_2, \cdots, \mathrm{Y}_p]\}N, \tag{3. 1}$$

and

$$\mathbf{y}(Np\times 1)=[\mathrm{Y}_1', \mathrm{Y}_2', \cdots, \mathrm{Y}_p']', \tag{3. 2}$$

then, under the model (2. 1), $\mathbf{y}(Np\times 1)$ *is distributed in normal distribution* $\mathrm{N}[\xi(Np\times 1)$, $\mathbf{V}(Np\times Np)]$, *where*

$$\xi(p\times 1)=[\mu_1\mathbf{1}'(1\times N),\ \mu_2\mathbf{1}'(1\times N),\ \cdots,\ \mu_p\mathbf{1}'(1\times N)]', \tag{3. 3}$$

and

$$\mathbf{V}(Np\times Np)=\sum_{i=1}^{k+1}\mathbf{A}(N\times N)\otimes\Sigma_i(p\times p). \tag{3. 4}$$

PROOF. It is easily seen that the expectation of $\mathbf{y}(Np\times 1)$ is given by (3. 1). Now, in virtue of (2. 1), the model equation of $\mathrm{Y}_l(N\times 1)$ is given by

$$\mathrm{Y}_l(N\times 1)=\mu_l\mathbf{1}(N\times 1)+\sum_{i=1}^{k}\mathbf{X}_i(N\times m_i)\beta_{il}(m_i\times 1)+\mathbf{X}_{k+1}(N\times N)\beta_{k+1,l}(N\times 1), \quad l=1,2,\cdots p, \tag{3. 5}$$

where β_{il} is the l-th column vector of $\mathbf{B}_i$ for $i=1,2,\cdots,\ k+1$,
and we have

$$\begin{aligned} E[\mathrm{Y}_l(N\times 1)\mathrm{Y}_s'(1\times N)] &= E[(\mu_l\mathbf{1}+\sum_{i=1}^{k+1}\mathbf{X}_i\beta_{il})(\mu_s\mathbf{1}+\sum_{i=1}^{k+1}\mathbf{X}_i\beta_{is})'] \\ &=\mu_l\mu_s\mathbf{E}(N\times N)+E(\sum_{i=1}^{k+1}\mathbf{X}_i\beta_{il}\beta_{is}'\mathbf{X}_i) \\ &=\mu_l\mu_s\mathbf{E}(N\times N)+\sum_{i=1}^{k+1}\mathbf{X}_i(\sigma_{ls}^{(i)}\mathbf{I}(N\times N))\mathbf{X}_i' \\ &=\mu_l\mu_s\mathbf{E}(N\times N)+\sum_{i=1}^{k+1}\sigma_{ls}^{(i)}\mathbf{A}_i(N\times N), \end{aligned} \tag{3. 6}$$

where $\sigma_{ls}^{(i)}$ is the (l,s)-element of Σ_i.

On the other hand, considering that

$$\begin{aligned} E[\mathbf{y}(Np\times 1)]E[\mathbf{y}'(Np\times 1)] &=\mathbf{E}(N\times N)\otimes\begin{pmatrix} \mu_1^2 & \mu_1\mu_2 & \mu_1\mu_3\cdots\cdots & \mu_1\mu_p \\ \mu_2\mu_1 & \mu_2^2 & \mu_2\mu_3\cdots\cdots & \mu_2\mu_p \\ \vdots & & & \vdots \\ \mu_p\mu_1 & \mu_p\mu_2 & \mu_p\mu_3\cdots\cdots & \mu_p^2 \end{pmatrix} \\ &=\mathbf{E}(N\times N)\otimes(\mathbf{B}_0'\mathbf{B}_0), \end{aligned} \tag{3. 7}$$

we obtain (3. 2).

Now we shall set up some combinations of the following assumptions in certain sections of this paper.

ASSUMPTION (I) $\mathbf{A}_i$'s $(i=0,1,2,\cdots,k+1)$ *commute in pairs.*

ASSUMPTION (II) *The elements of* $\mathbf{A}_i$ *are equal to 0 or 1 and it holds that* $\mathbf{1}'(1\times N)\mathbf{A}_i=r_i\mathbf{1}'(1\times N)$.

ASSUMPTION (III) $\mathbf{A}_i$'s $(i=0,1,\cdots,k+1)$ *are linearly independent.*

ASSUMPTION (IV) *The diagonal elements of* Σ_i *are equal to each other, while other elements of* Σ_i *are also equal among themselves for* $i=1,2,\cdots,\ k+1,$ *hence* Σ_i *has the form*

$$\Sigma_i=\begin{pmatrix} \sigma_i^2 & & & \\ & \sigma_i^2 & & \tau_i \\ & & \ddots & \\ \tau_i & & & \sigma_i^2 \end{pmatrix}, \quad (i=1,2,\cdots,k+1). \tag{3. 8}$$

The Assumptions (I), (II) and (III) are concerned with the layout of experiments. The models of the experimental designs with equal numbers in the subclasses, which include the r-way layout models with or without interaction, the r-fold nested classfication models, the split-plot models, etc., satisfy these assumptions. For in these experimental designs all $\mathbf{A}_i$ are expressed in the form of the Kronecker product of several numbers of $\mathbf{E}$ and $\mathbf{I}$ such that the dimensions of the corresponding component matrices $\mathbf{E}$ or $\mathbf{I}$ are equal among themselves, and all $\mathbf{A}_i$ are different from each other.

The models of the experimental designs with unequal numbers in the subclass do not satisfy these assumptions. For example, in the 2-way layout without interaction such that the treatment combinations are given by (12), (21), (22), (33), (34), (43) and (44), $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ can be written as

$$
(3.\ 9) \qquad \mathbf{A}_1=\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&1&0&0&0&0\\ 0&1&1&0&0&0&0\\ 0&0&0&1&1&0&0\\ 0&0&0&1&1&0&0\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&1&1 \end{pmatrix}, \qquad \mathbf{A}_2=\begin{pmatrix} 1&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 1&0&1&0&0&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1 \end{pmatrix}, \qquad \mathbf{A}_3=\mathbf{I}(7\times 7),
$$

which implies that this is the case. The models of B.I.B. designs satisfy the Assumption (II), but not the Assumption (I).

THEOREM 3. 2. *Under the Assumptions* (I), (II), *and* (IV), *there exists an orthogonal transformation which transforms* $\mathbf{V}$ *given by (3. 4) into a diagonal matrix.*

PROOF. In virtue of the Assumptoin (I) and the symmetricity of $\mathbf{A}_i$, there exists an orthogonal matrix which diagonalizes $\mathbf{A}_0, \mathbf{A}_1, \cdots\cdots, \mathbf{A}_{k+1}$. And under the Assumption (II) this can be realized by an orthogonal matrix $\mathbf{P}$, which is defined in Section 2. Therefore let us consider such a particular matrix $\mathbf{P}$. Let $\mathbf{Q}$ be any orthogonal matrix defined in Section 2.

Then, since Σ_i is expressed as follows

$$
(3.\ 10) \qquad \Sigma_i=\tau_i\mathbf{E}(p\times p)+(\sigma_i^2-\tau_i)\,\mathbf{I}(p\times p),
$$

we have

$$
\begin{aligned}
(3.\ 11) \quad & (\mathbf{P}(N\times N)\otimes\mathbf{Q}(p\times p))\mathbf{V}(\mathbf{P}(N\times N)\otimes\mathbf{Q}(p\times p))' \\
&=\sum_{i=1}^{k+1}\{p\tau_i(\mathbf{P}\mathbf{A}_i\mathbf{P}')\otimes\mathbf{H}(p\times p)+(\sigma_i^2-\tau_i)(\mathbf{P}\mathbf{A}_i\mathbf{P}')\otimes\mathbf{I}(p\times p)\} \\
&=\sum_{i=1}^{k+1}\begin{pmatrix} p\tau_i\Lambda_i(N\times N) & \mathbf{0} & & \\ & \mathbf{0} & & \\ & & \mathbf{0} & \\ \mathbf{0} & & & \ddots \\ & & & & \mathbf{0} \end{pmatrix}+\begin{pmatrix} (\sigma_i^2-\tau_i)\Lambda_i(N\times N) & & & \\ & (\sigma_i^2-\tau_i)\Lambda_i(N\times N) & & \mathbf{0} \\ \mathbf{0} & & \ddots & \\ & & & (\sigma_i^2-\tau_i)\Lambda_i(N\times N) \end{pmatrix}
\end{aligned}
$$

$$
=\begin{pmatrix}
\sum_{i=1}^{k+1}\{\sigma_i^2+(p-1)\tau_i\}\Lambda_i(N\times N) & & & & \\
& \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i(N\times N) & & \mathbf{0} & \\
& & \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i(N\times N) & & \\
& \mathbf{0} & & \ddots & \\
& & & & \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i(N\times N)
\end{pmatrix},
$$

where each $\Lambda_i(N\times N)=\mathbf{P}\mathbf{A}_i\mathbf{P}'$ is a diagonal matrix for $i=1,2,\cdots,k+1$, which completes the proof. q. e. d..

The direct consequence of Theorem 3. 2. is given by

COROLLARY 3. 1. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be be the orthogonal matrices defined in Section 2. Then we have*

$$
\text{(3. 12)}\qquad ((\mathbf{P}\otimes\mathbf{Q})\mathbf{V}(\mathbf{P}\otimes\mathbf{Q})')^{-1}=\begin{pmatrix}
\mathbf{D}_1(N\times N) & & & & \mathbf{0} \\
& \mathbf{D}_2(N\times N) & & & \\
& & \mathbf{D}_2(N\times N) & & \\
& & & \ddots & \\
\mathbf{0} & & & & \mathbf{D}_2(N\times N)
\end{pmatrix},
$$

where

$$
\mathbf{D}_1=\begin{pmatrix}
\frac{1}{g_1} & & & 0 \\
& \times & & \\
0 & & \times & \\
& & & \ddots \\
& & & & \times
\end{pmatrix}\ \textit{and}\ \ \mathbf{D}_2=\begin{pmatrix}
\frac{1}{g_2} & & & 0 \\
& \times & & \\
0 & & \times & \\
& & & \ddots \\
& & & & \times
\end{pmatrix},
$$

and where g_1 *and* g_2 *are the distinct elements of the first row in the first column on the matrices* $\sum_{i=1}^{k+1}\{\sigma_i^2+(p-1)\tau_i\}\Lambda_i$ *and* $\sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i$, *respectively.*

4. Estimability Theorem.

We shall define the notion of estimability for Σ_i in our concern and then refer to the necessary and the sufficient conditions for Σ_i to be estimable.

Definition. Under the Assumption (II), the parameter matrix Σ_i is said to be estimable if and only if the quadratic forms $\mathbf{y}'\mathbf{G}_i\mathbf{y}$ and $\mathbf{y}'\mathbf{M}_i\mathbf{y}$ exist such that $E[\mathbf{y}'\mathbf{G}_i\mathbf{y}]=\sigma_i^2$ and $E[\mathbf{y}'\mathbf{M}_i\mathbf{y}]=\tau_i$.

Now we observe

THEOREM 4. 1. *Under the Assumption* (IV) *a necessary condition for* Σ_i's $(i=1,2,\cdots,k+1)$ *to be estimable is that* $\mathbf{A}_1,\cdots\cdots,\mathbf{A}_{k+1}$ *are linearly independent.*

PROOF. Let

$$\mathbf{B}_i=[\beta_{i1},\ \beta_{i2},\ \cdots\cdots,\ \beta_{ip}],$$

and

$$\tilde{\mathbf{B}}_i=[\beta_{i1}',\ \beta_{i2}'\cdots\cdots,\ \beta_{ip}']', \qquad (i=1,2,\cdots,k+1).$$

Then it holds that

$$\mathbf{y}(Np\times 1)=\mathbf{1}(N\times N)\otimes\mathbf{B}_0+\sum_{i=1}^{k+1}(\mathbf{X}_i\otimes\mathbf{I}(p\times p))\tilde{\mathbf{B}}_i. \tag{4. 1}$$

Let Σ_i's $(i=1,2,\cdots,k+1)$ be estimable. Then there exist $\mathbf{G}_u$'s and $\mathbf{M}_u$'s such that

$$E[\mathbf{y}'\mathbf{G}_u\mathbf{y}]=\sigma_u^2, \qquad (u=1,2,\cdots,k+1), \tag{4. 2}$$

and

$$E[\mathbf{y}'\mathbf{M}_u\mathbf{y}]=\tau_u, \qquad (u=1,2,\cdots,k+1). \tag{4. 3}$$

The left hand side of (3. 11) is expressed as follows.

$$\begin{aligned}
(4.\ 4)\qquad & E[\{\mathbf{1}\otimes\mathbf{B}_0+\sum_{i=1}^{k+1}(\mathbf{X}_i\otimes\mathbf{I})\tilde{\mathbf{B}}_i\}'\mathbf{G}_u\{\mathbf{1}\otimes\mathbf{B}_0+\sum_{i=1}^{k+1}(\mathbf{X}_i\otimes\mathbf{I})\tilde{\mathbf{B}}_i\}]\\
&=E[\{\sum_{i=1}^{k+1}\tilde{\mathbf{B}}_i'(\mathbf{X}_i\otimes\mathbf{I})'\}\mathbf{G}_u\{\sum_{i=1}^{k+1}(\mathbf{X}_i\otimes\mathbf{I})\tilde{\mathbf{B}}_i\}]+(\mathbf{1}'\otimes\mathbf{B}_0')\mathbf{G}_u(\mathbf{1}\otimes\mathbf{B}_0)\\
&=E\sum_{i=1}^{k+1}\mathrm{tr}[(\mathbf{X}_i\otimes\mathbf{I})'\mathbf{G}_u(\mathbf{X}_i\otimes\mathbf{I})\tilde{\mathbf{B}}_i\tilde{\mathbf{B}}_i']+\mathrm{tr}[\mathbf{G}_u(\mathbf{1}\mathbf{1}'\otimes\mathbf{B}_0\mathbf{B}_0')]\\
&=\sum_{i=1}^{k+1}\mathrm{tr}[(\mathbf{X}_i\otimes\mathbf{I})'\mathbf{G}_u(\mathbf{X}_i\otimes\mathbf{I})\{\tau_i\mathbf{I}\otimes\mathbf{E}+(\sigma_i^2-\tau_i)\mathbf{I}\otimes\mathbf{I}\}]\\
&\qquad+\mathrm{tr}[\mathbf{G}_u(\mathbf{E}\otimes\mathbf{B}_0\mathbf{B}_0')]\\
&=\sum_{i=1}^{k+1}\tau_i\mathrm{tr}[\{(\mathbf{X}_i\mathbf{X}_i')\otimes\mathbf{E}\}\mathbf{G}_u]+\sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\mathrm{tr}[\{(\mathbf{X}_i\mathbf{X}_i')\otimes\mathbf{I}\}\mathbf{G}_u]\\
&\quad+\mathrm{tr}[\mathbf{G}_u(\mathbf{E}\otimes\mathbf{B}_0\mathbf{B}_0')].
\end{aligned}$$

Therefore it follows that, in virtue of (4. 11),

$$\begin{aligned}
(4.\ 5)\qquad & \sum_{i=1}^{k+1}\sigma_i^2\mathrm{tr}[(\mathbf{A}_i\otimes\mathbf{I})\mathbf{G}_u]+\sum_{i=1}^{k+1}\tau_i\mathrm{tr}[\{\mathbf{A}_i\otimes(\mathbf{E}-\mathbf{I})\}\mathbf{G}_u]+\mathrm{tr}[\mathbf{G}_u(\mathbf{E}\otimes\mathbf{B}_0\mathbf{B}_0')]\\
&=\sigma_u^2, \qquad (u=1,2,\cdots,k+1).
\end{aligned}$$

Similarly, it holds that

$$\sum_{i=1}^{k+1}\sigma_i^2\mathrm{tr}[(\mathbf{A}_i\otimes\mathbf{I})\mathbf{M}_u]+\sum_{i=1}^{k+1}\tau_i\mathrm{tr}[\{\mathbf{A}_i\otimes(\mathbf{E}-\mathbf{I})\}\mathbf{M}_u]+\mathrm{tr}[\mathbf{M}_u(\mathbf{E}\otimes\mathbf{B}_0\mathbf{B}_0')] \tag{4. 6}$$

$$=\tau_u, \qquad (u=1,2,\cdots,k+1).$$

Since the equation (4. 5) holds true for all non-negative values of all σ_i^2, we obtain that

$$\mathrm{tr}[(\mathbf{A}_i\otimes\mathbf{I})\mathbf{G}_u]=\begin{cases}0 & \text{when } i\neq u,\\ 1 & \text{when } i=u,\end{cases} \qquad (u=1,2,\cdots,k+1).$$

In order to show the linear independency of $\mathbf{A}_i$'s ($i=1,\cdots,k+1$), let $c_1,\cdots,c_{k+1}$ be any set of constants such that

$$\sum_{i=1}^{k+1} c_i(\mathbf{A}_i\otimes\mathbf{I})=\mathbf{O}.$$

Then, since it holds that

$$\sum_{i=1}^{k+1} c_i\,\mathrm{tr}[(\mathbf{A}_i\otimes\mathbf{I})\mathbf{G}_u]=c_u, \qquad (u=1,2,\cdots,k+1),$$

we have that

$$\mathrm{tr}\,[\mathbf{G}_u\{\sum_{i=1}^{k+1} c_i(\mathbf{A}_i\otimes\mathbf{I})\}]=\sum_{i=1}^{k+1} c_i\,\mathrm{tr}\,[(\mathbf{A}_i\otimes\mathbf{I})\mathbf{G}_u]=c_u,$$

which implies that $c_i=0$ for $i=1,2,\cdots,k+1$.

Therefore $\mathbf{A}_1\otimes\mathbf{I},\cdots,\mathbf{A}_{k+1}\otimes\mathbf{I}$ are linearly independent, which implies that $\mathbf{A}_1,\cdots,\mathbf{A}_{k+1}$ are linearly independent.

The same result can be obtained from (4. 6) in the similar way.

THEOREM 4. 2. *Under the Assumption* (IV) *a sufficient condition for* Σ_i's ($i=1,2,\cdots,k+1$) *to be estimable is that* $\mathbf{A}_0,\mathbf{A}_1,\cdots,\mathbf{A}_{k+1}$ *are linearly independent.*

PROOF. Let us assume that $\mathbf{A}_0,\mathbf{A}_1,\cdots,\mathbf{A}_{k+1}$ are linearly independent. Then we have

$$E[Y_lY_s']=\mu_l\mu_s\mathbf{A}_0+\sum_{i=1}^{k+1}\tau_i\mathbf{A}_i, \text{ for } \quad l\neq s, \tag{4. 7}$$

and

$$E[Y_lY_l']=\mu_l^2\mathbf{A}_0+\sum_{i=1}^{k+1}\sigma_i^2\mathbf{A}_i. \tag{4. 8}$$

Now let $\omega_{\alpha\beta}=y_\alpha^{(l)}y_\beta^{(l)}$ where $y_\alpha^{(l)}$ is the element in the α-th row of Y_l and let the vector $\mathrm{W}(\frac{n(n+1)}{2}\times 1)$ be defined by

$$\mathrm{W}(\tfrac{n(n+1)}{2}\times 1)=[\omega_{11},\omega_{12},\cdots,\omega_{1N},\omega_{22},\cdots,\omega_{2N},\cdots,\omega_{pp}]'.$$

And let the (α,β)-th element of $\mathbf{A}_i$ be $a_{\alpha\beta}^{(i)}$ and let the vector $\mathfrak{A}_i$ be defined by

$$\mathfrak{A}_i(\tfrac{n(n+1)}{2}\times 1)=[a_{11}^{(i)},a_{12}^{(i)},\cdots,a_{1N}^{(i)},a_{22}^{(i)},\cdots,a_{2N}^{(i)},\cdots,a_{NN}^{(i)}]'.$$

Then it holds that

$$E(\mathrm{W})=\mu_i^2\mathfrak{A}_0+\sum_{i=1}^{k+1}\sigma_i^2\mathfrak{A}_i. \tag{4. 9}$$

Since $\mathbf{A}_0, \mathbf{A}_1, \cdots, \mathbf{A}_{k+1}$ are linearly independent, $\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_{k+1}$ are also linearly independent.

Putting $\mathfrak{A}(\frac{n(n+1)}{2}\times(k+2))=[\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_{k+1}]$ and $\mathfrak{C}((k+2)\times 1)=[\mu_i^2, \sigma_1^2, \cdots, \sigma_{k+1}^2]'$, (4.9) can be written as $E(\mathrm{W})=\mathfrak{A}\mathfrak{C}$. Since $\mathfrak{A}$ has rank $k+2$, $\mathfrak{C}$ is given in the form $\mathfrak{C}=\mathbf{L}E(\mathrm{W}^*)$, where W^* is a subvector of W, in virtue of (4. 8). Thus all σ_i^2's are estimable.

Further, from (4. 7) it is showed that all τ_i's are estimable in the similar way.

q. e. d.

5. Characteristic roots of the variance matrix.

In this section we shall discuss some of the properties of the characteristic roots, which play an important role in the following sections.

THEOREM 5. 1. *Under the Assumptions* (I), (II), (III) *and* (IV), *the number of the distinct characteristic roots of the matrix* $\mathbf{V}$ *is not less than* $2k+2$.

PROOF. In the proof of Theorem 3. 2, we showed that $\mathbf{V}$ was transformed to the diagonal matrix. And all $\{\sigma_i^2+(p-1)\tau_i\}$ and all $(\sigma_i^2-\tau_i)$ are functionally independent. Therefore it can be shown that the number of the distinct elements of

$$\begin{bmatrix} \sum_{i=1}^{k+1}\{\sigma_i^2+(p-1)\tau_i\}\Lambda_i & \mathbf{0} \\ \mathbf{0} & \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i \end{bmatrix}$$

is not less than $2k+2$, along the same line as that of Theorem 3 in [9], which establishes the proof. q. e. d.

Secondly, we shall show

THEOREM 5. 2. *Under the Assumptions* (I), (II), (III) *and* (IV), *the* $2(k+1)$ *of the distinct characteristic roots of* $\mathbf{V}$ *are functionally independent.*

PROOF. Consider the last form in (3. 8). Let Λ^* and Λ_i^* be defined as the vectors of the diagonal elements of the diagonal matrices $(\mathbf{P}\otimes\mathbf{Q})\mathbf{V}(\mathbf{P}\otimes\mathbf{Q})'$ and Λ_i respectively. Then it holds that

$$\Lambda^*(Np\times 1)=\begin{bmatrix} \sum_{i=1}^{k+1}\{\sigma_i^2+(p-1)\tau_i\}\,\Lambda_i^*(N\times 1) \\ \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i^*(N\times 1) \\ \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i^*(N\times 1) \\ \vdots \\ \vdots \\ \sum_{i=1}^{k+1}(\sigma_i^2-\tau_i)\Lambda_i^*(N\times 1) \end{bmatrix} \tag{5. 1}$$

Now we have

$$\sum_{i=1}^{k+1} \{\sigma_i^2+(p-1)\tau_i\}\Lambda_i^*(N\times 1)=[\Lambda_1^*, \Lambda_2^*, \cdots, \Lambda_{k+1}^*] \left| \begin{array}{c} \sigma_1^2+(p-1)\tau_1 \\ \sigma_2^2+(p-1)\tau_2 \\ \vdots \\ \sigma_{k+1}^2+(p-1)\tau_{k+1} \end{array} \right| .$$

Since $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_{k+1}$ are linearly independent, $\Lambda_1^*, \Lambda_2^*, \cdots, \Lambda_{k+1}^*$ are linearly independent, which implies that the rank of the matrix $[\Lambda_1^*, \Lambda_2^*, \cdots, \Lambda_{k+1}^*]$ is equal to $k+1$. But $\{\sigma_i^2+(p-1)\tau_i\}$'s are functionally independent. Therefore $\sum_{i=1}^{k+1} \{\sigma_i^2+(p-1)\tau_i\} \Lambda_i^*$ has $k+1$ functionally independent elements.

Similarly, it holds that $\sum_{i=1}^{k+1} (\sigma_i^2-\tau_i)\Lambda_i^*$ has also $k+1$ functionally independent elements.

Since all $\{\sigma_i^2+(p-1)\tau_i\}$ and all $(\sigma_i^2-\tau_i)$ are functionally independent, the vector Λ^* has $2(k+1)$ functionally independent elements, which establishes the theorem. q. e. d.

6. Complete sufficient set.

In this section we shall derive the sufficient statistics in the model defined in Section 2 under the assumptions (I), (II), (III) and (IV) and then we shall discuss their distributions.

Now let us consider the quadratic form

$$Z=(\mathrm{y}-(\mathbf{1}\otimes\mathbf{B}_0))'\mathbf{V}^{-1}(\mathrm{y}-(\mathbf{1}\otimes\mathbf{B}_0)), \tag{6. 1}$$

and let introduce an orthogonal transformation $\mathbf{P}\otimes\mathbf{Q}$, where $\mathbf{P}$ and $\mathbf{Q}$ are defined in Section 2. Then, in virtue of Corollary 3. 1, Z is given by

$$\begin{aligned} Z=&[(\mathbf{P}\otimes\mathbf{Q})\mathrm{y}-(\mathbf{P}\otimes\mathbf{Q})(\mathbf{1}\otimes\mathbf{B}_0)]'[(\mathbf{P}\otimes\mathbf{Q})\mathbf{V}(\mathbf{P}\otimes\mathbf{Q})']^{-1} \\ &[(\mathbf{P}\otimes\mathbf{Q})\mathrm{y}-(\mathbf{P}\otimes\mathbf{Q})(\mathbf{1}\otimes\mathbf{B}_0)] \\ =&\frac{1}{g_1}\{(P_1(1\times N)\otimes Q_1(1\times p))\mathrm{y}(Np\times 1)-\sqrt{N}\sum_{j=1}^{p} q_{1j}\mu_j\}^2 \\ &+\frac{1}{g_2}[\sum_{h=2}^{p}\{(P_1(1\times N)\otimes Q_h(1\times p))\mathrm{y}(Np\times 1)-\sqrt{N}\sum_{j=1}^{p} q_{hj}\mu_j\}^2] \\ &+\sum_{u=3}^{s}\frac{1}{g_u}\mathrm{y}'(1\times Np)\mathbf{R}_u'(Np\times m_u)\mathbf{R}_u(m_u\times Np)\mathrm{y}(Np\times 1), \end{aligned} \tag{6. 2}$$

where $P_i(1\times N)$ is the i-th row vector of $\mathbf{P}$, $\mathbf{Q}_j$ is the j-th row vector of $\mathbf{Q}$, g_u's ($u=1, 2, \cdots, s$) are the distinct characteristic roots of $\mathbf{V}$, each row vector of all $\mathbf{R}_u$ is equal to one of $p(N-1)$ row vectors $P_i\otimes Q_j$'s ($i=2, \cdots, N$; $j=1, \cdots, p$), all row vectors of all $\mathbf{R}_u$ are distinct from each other and n_u, the row dimension of $\mathbf{R}_u$, is equal to the multiplicity of the characteristic root g_u. From the last form of (6. 2) it is easily seen that a set of $p+s-2$ statistics $(P_1\otimes Q_j)\mathrm{y}$'s ($j=1, 2, \cdots, p$) and $\mathrm{y}'\mathbf{R}_u'\mathbf{R}_u\mathrm{y}$'s ($u$

$=3, 4, \cdots, s$) are the sufficient statistics for the family of distribution of all observations under our model.

Now we shall derive the distributions of these statistics. $(P_1 \otimes Q_1)\,\mathbf{y}$ is distributed as a univariate normal, whose mean is given by $E[(P_1 \otimes Q_1)\mathbf{y}] = (P_1 \otimes Q_1)(\mathbf{1} \otimes \mathbf{B}_0) = \sqrt{N}\sum_{j=1}^{p} q_{1j}\mu_j$ and the variance is given by

$$
\begin{aligned}
(6.\ 3) \qquad & E[\{(P_1 \otimes Q_1)\mathbf{y} - \sqrt{N}\sum_{j=1}^{p} q_{1j}\mu_j\}\{(P_1 \otimes Q_1)\mathbf{y} - \sqrt{N}\sum_{j=1}^{p} q_{1j}\mu_j\}'] \\
& = E[(P_1 \otimes Q_1)\mathbf{y}\mathbf{y}'(P_1 \otimes Q_1)'] - N\Big(\sum_{j=1}^{p} q_{1j}\mu_j\Big)^2 \\
& = (P_1 \otimes Q_1)\mathbf{V}(P_1 \otimes Q_1)' \\
& = g_1.
\end{aligned}
$$

Similarly, it is seen that $(P_1 \otimes Q_j)\,\mathbf{y}$ is distributed as a univariate normal with mean $(P_1 \otimes Q_j)(\mathbf{1} \otimes \mathbf{B}_0) = \sqrt{N}\sum_{h=1}^{p} q_{jh}\mu_h$ and variance $(P_1 \otimes Q_j)\mathbf{V}(P_1 \otimes Q_j)' = g_2$ for $j=2, \cdots, p$.

On the other hand we obtain the following;

(A) $\mathbf{R}_u'\mathbf{R}_u\mathbf{V}/g_u$ is idempotent since it holds that

$$\mathbf{R}_u'(\mathbf{R}_u\mathbf{V}\mathbf{R}_u')\mathbf{R}_u\mathbf{V}/g_u^2 = \mathbf{R}_u' g_u \mathbf{I}(n_u \times n_u)\mathbf{R}_u\mathbf{V}/g_u^2 = \mathbf{R}_u'\mathbf{R}_u/g_u.$$

(B) $E[\mathbf{R}_u\mathbf{y}] = (P_l \otimes Q_m)(\mathbf{1} \otimes \mathbf{B}_0) = \mathbf{0}$ for $l \neq 1$, and $\mathrm{Var}[\mathbf{R}_u\mathbf{y}] = g_u$.

(C) rank $(\mathbf{R}_u'\mathbf{R}_u\mathbf{V}) = n_u$.

(D) $\mathbf{R}_u\mathbf{V}\mathbf{R}_v' = \mathbf{0}$ for $u \neq v$, since $(P_i \otimes Q_j)\mathbf{V}(P_k \otimes Q_l)'$

is not zero if and only if $(i, j) = (k, l)$.

The above results show that all $\mathbf{y}\mathbf{R}_u'\mathbf{R}_u\mathbf{y}$ are distributed independently to each other in central chi-square distributions with n_u degrees of freedom and also independently to all $(P_1 \otimes Q_j)\,\mathbf{y}$.

This consideration will be summarized in the following.

THEOREM 6. 1. *In addition to the Assumptions* (I), (II), (III) *and* (IV), *let us assume that* $\mathbf{V}$ *has s distinct characteristic roots. Then the sufficient statistics for the family of the distribution of all observations are given by* $(P_1 \otimes Q_j)\,\mathbf{y}$'s $(j=1, 2, \cdots, p)$ *and* $\mathbf{y}'\mathbf{R}_u'\mathbf{R}_u\mathbf{y}$'s $(u=3, 4, \cdots, s)$.

Lastly in this section, we shall add the following theorem which seems to be much useful in seeking for the minimum variance unbiased estimate of Σ_i.

THEOREM 6. 2. *In addition to the Assumptions* (I), (II), (III) *and* (IV), *let us add that* $\mathbf{V}$ *has* $2k+1$ *distinct characteristic roots. Then the* $(2k+1+p)$ *statistics* $(P_1 \otimes Q_j)\,\mathbf{y}$'s $(j=1, 2, \cdots, p)$ *and* $\mathbf{y}'\mathbf{R}_u'\mathbf{R}_u\mathbf{y}$'s $(u=3, 4, \cdots, 2k+4)$ *form a complete sufficient set for the family of the distribution of all observations in our concern.*

PROOF. If $\mathbf{V}$ has $2k+4$ distinct characteristic roots, then the quadratic form in exponent (6. 2) is equal to

$$Z=\frac{1}{g_1}\{(P_1(1\times N)\otimes Q_1(1\times p))\mathbf{y}(Np\times 1)\}^2+\frac{1}{g_2}\sum_{h=2}^{p}\{(P_1(1\times N)\otimes Q_h(1\times p))\mathbf{y}(Np\times 1)\}^2+\sum_{u=3}^{2k+4}\frac{1}{g_u}\mathbf{y}'\mathbf{R}_u\mathbf{R}_u\mathbf{y}$$
$$-\frac{2\sqrt{N}\left(\sum_{j=1}^{p}g_{1j}\mu_j\right)}{g_1}(P_1(1\times N)\otimes Q_1(1\times p))\,\mathbf{y}(Np\times 1)$$
$$-\frac{2\sqrt{N}}{g_2}\sum_{h=2}^{p}\left(\sum_{j=1}^{p}g_{hj}\mu_j\right)(P_1(1\times N)\otimes Q_h(1\times p))\,\mathbf{y}(Np\times 1)$$
$$+\varphi(\mu_1,\mu_2,\cdots,\mu_p), \tag{6. 4}$$

where $\varphi(\mu_1, \mu_2, \cdots, \mu_p)$ is the function of $\mu_1, \mu_2, \cdots, \mu_p$.

Now we shall consider the transformations of the original parameters and the sufficient statistics such that

$$\theta_1=-\frac{2\sqrt{N}\left(\sum_{j=1}^{p}q_{1j}\mu_j\right)}{g_1}, \tag{6. 5}$$

$$\theta_h=-\frac{2\sqrt{N}\left(\sum_{j=1}^{p}q_{hj}\mu_j\right)}{g_2}, \qquad (h=2,3,\cdots,p), \tag{6. 6}$$

$$\theta_k=\frac{1}{g_{k-(p-2)}}, \qquad (k=p+1,p+2,\cdots,2k+2+p), \tag{6. 7}$$

$$U_1=(P_1(1\times N)\otimes Q_1(1\times p))\,\mathbf{y}(Np\times 1), \tag{6. 8}$$

$$U_h=(P_1(1\times h)\otimes Q_h(1\times p))\,\mathbf{y}(Np\times 1), \qquad (h=2,3,\cdots,p), \tag{6. 9}$$

$$U_k=\mathbf{y}'\mathbf{R}_{k-(p-2)}\mathbf{R}_{k-(p-2)}\mathbf{y}, \qquad (k=p+1,p+2,\cdots,2k+2+p). \tag{6. 10}$$

Then we should notice that the transformations (6. 5), $\cdots$, (6. 10) from $\tau=(\mu_1, \mu_2, \cdots, \mu_p, \sigma_1^2, \sigma_2^2, \cdots, \sigma_{k+1}^2, \tau_1, \tau_2, \cdots, \tau_{k+1})$ to $\theta=(\theta_1, \theta_2, \cdots, \theta_{2k+2+p})$ is one-to-one, because of the orthogonality of $\mathbf{Q}$ and the functional independency among g_u's $(u=3, 4, \cdots, 2k+4)$, which is proved in Theorem 5. 2. Consequently it can be seen also that g_1 and g_2 are the functions of the new parameters θ_k's $(k=p+1, p+2, \cdots, 2k+2+p)$.

Thus, under the new parameters, the quadratic form in exponent is given by

$$\sum_{i=1}^{2k+2+p}\theta_iU_i+g_1(\theta_{p+1},\cdots,\theta_{2k+2+p})U_1^2+g_2(\theta_{p+1},\cdots,\theta_{2k+2+p})\sum_{j=2}^{p}U_j^2+\varphi(\theta_1,\cdots,\theta_p), \tag{6. 11}$$

where $g_1(\theta_{p+1}, \cdots, \theta_{2k+2+p})$ and $g_2(\theta_{p+1}, \cdots, \theta_{2k+2+p})$ are the function of θ_k's $(k=p+1, p+2, \cdots, 2k+2+p)$.

The theorem is completed by applying the result of Lemma 4. 8 in our previous paper [4] to (6. 11). q. e. d.

7. Examples.

EXAMPLE 1. Concider the p-variate complete 2-way layout model without interaction in which the levels of all treatments are equal to three, and Assumption (IV) is satisfied. Then the design matrix is given by

$$(7.1)\qquad \left(\begin{array}{ccccccc|c} 1 & 1 & 0 & 0 & 1 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & \mathbf{I}(9\times 9) \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & \end{array}\right),$$

and $\mathbf{A}_i$'s can be written as follows: $\mathbf{A}_1=\mathbf{E}(3\times3)\otimes\mathbf{I}(3\times3)$, $\mathbf{A}_2=\mathbf{I}(3\times3)\otimes\mathbf{E}(3\times3)$, $\mathbf{A}_3=\mathbf{I}(3\times3)\otimes\mathbf{I}(3\times3)$. Obviously Asumptions (I), (II) and (III) are satisfied in this case.

Moreover, hence $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ are transformed to $\Lambda_1=3\mathbf{H}(3\times3)\otimes\mathbf{I}(3\times3)$, $\Lambda_2=3\mathbf{I}(3\times3)\otimes\mathbf{H}(3\times3)$ and $\Lambda_3=\mathbf{I}(3\times3)\otimes\mathbf{I}(3\times3)$ respectively, it is showed that for any c_1, c_2 and c_3.

$$(7.2)\qquad \sum_{i=1}^{3} c_i\Lambda=\begin{pmatrix} 3c_1+3c_2+c_3 & & & & & & & & \\ & 3c_2+c_3 & & & & & & & \\ & & 3c_2+c_3 & & & & & 0 & \\ & & & 3c_1+c_3 & & & & & \\ & & & & c_3 & & & & \\ & & & & & c_3 & & & \\ & & & & & & 3c_1+c_3 & & \\ & 0 & & & & & & c_3 & \\ & & & & & & & & c_3 \end{pmatrix}.$$

Therefore, in virtue of the last form of (3. 8), $\mathbf{V}$ has eight distinct characteristic roots $3\alpha_1+3\alpha_2+\alpha_3$, $3\alpha_1+\alpha_3$, $3\alpha_2+\alpha_3$, α_3, $3\beta_1+3\beta_2+\beta_3$, $3\beta_1+\beta_3$, $3\beta_2+\beta_3$, β_3, where $\alpha_i=\sigma_i^2+(p-1)\tau_i$ and $\beta_i=\sigma_i^2-\tau_i$.

Thus, from Theorem 6. 2, it can be seen that there exist the unique minimum variance unbiased estimates of Σ_i's $(i=1, 2, 3)$.

EXAMPLE 2. Consider the incomplete 2-way layout model without interaction in which the treatment combinations are given by (11), (12), (21), (22), (33), (34), (43) and (44) and Assumption (IV) is satisfied.
Then $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ can be written as follows: $\mathbf{A}_1=\mathbf{E}(2\times2)\otimes\mathbf{I}(2\times2)\otimes\mathbf{I}(2\times2)$, $\mathbf{A}_2=\mathbf{I}(2\times2)\otimes\mathbf{E}(2\times2)\otimes\mathbf{I}(2\times2)$, $\mathbf{A}_3=\mathbf{I}(8\times8)$. And these satisfy the Assumptions (I), (II) and (III).

Hence $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ are transformed to $\Lambda_1=2\mathbf{H}(2\times2)\otimes\mathbf{I}(2\times2)\otimes\mathbf{I}(2\times2)$, $\Lambda_2=2\mathbf{I}(2\times2)\otimes\mathbf{H}(2\times2)\otimes\mathbf{I}(2\times2)$ and $\Lambda_3=\mathbf{I}(8\times8)$, eight distinct characteristic roots of $\mathbf{V}$ are given by $2\alpha_1+2\alpha_2+\alpha_3$, $2\alpha_1+\alpha_3$, $2\alpha_2+\alpha_3$, α_3, $2\beta_1+2\beta_2+\beta_3$, $2\beta_1+\beta_3$, $2\beta_2+\beta_3$, β_3 where α_i and β_i are defined in Example 1.

Therefore there exist the unique munimun variance unbiased estimates of Σ_i's $(i=1, 2, 3)$.

8. Remark.

After treating the multivariate random effect model under the Assumptions (I), (II), (III) and (IV) in this paper, there naturally arises the corresponding problem for more general situation without the Assumption (IV), which is important for general application of random models. The similar problems for the case of mixed model are worthwhile to be discussed in detail. The author should like to have another ocasion to discuss some of these problems.

9. Acknowledgement.

The author would like to express his hearty thanks to Professor T. Kitagawa of the mathematical institute, Kyushu University for his helpful suggestions and criticisms while this paper was being prepared. The author is also grateful to Dr. A. Kudo for his encouragements and criticisms.

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ON REPRESENTATIONS OF JORDAN TRIPLE SYSTEMS

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(Received Nov. 25, 1961)

The purpose of this paper is to abstract the notion of a Jordan triple system which has been introduced by Jacobson in [7][1] and to study its representation. Duffin [4] and Kemmer [11] first considered such system, β-matrices, for describing the meson and these matrices were studied by many authors. On the other hand, a Jordan triple system is an example of the so called affine structure which appeared in the study of 0-connection on the group space by E. Cartan [2], therefore it seems that it is appropriate to study this system from the Lie triple system-like stand point. Garnir [6] and Jacobson [7] have already used this fact and the latter obtained also many general properties of Jordan triple systems. But it seems that contrary to the cases of Jordan algebras and Lie triple systems, the abstract studies of Jordan triple systems are relatively few.

We give the definition of Jordan triple system of type I and of type II on the notion of triple derivations. Following Eilenberg [5], we define the generalized representations for these systems. Next, for the Jordan triple system J_{I} of type I we consider a cohomology group $H^n(J_{\mathrm{I}}, V)$ which is associated with the representation by an analogous way to the method of Chevalley and Eilenberg [3]. Let J_{I}^* be an associated Lie triple system of J_{I}, then $H^n(J_{\mathrm{I}}, V)$ is mapped homomorphically into the cohomology group $H^n(J_{\mathrm{I}}^*, V)$ of J_{I}^*. Also, for the Jordan triple system J_{II} of type II we define a cohomology group $H^n(J_{\mathrm{II}}, V)$ which is associated with the representation. Let J_{II} be an associated Jordan triple system of type II of J_{I}, then $H^n(J_{\mathrm{II}}, V)$ is isomorphic to $H^n(J_{\mathrm{I}}, V)$. Throughout this paper, we assume that the base field is of characteristic 0 and the dimension of Jordan triple system is finite.

1. Basic definitions. We begin with the abstraction of the notion of subspaces of associative algebras which are closed relative to the Jordan triple product $\{a\{bc\}\}$, where $\{ab\} = ab + ba$, in two ways.

DEFINITION 1.1. A *Jordan triple system* (J. t. s.) J_{I} *of type* I is a vector space over a field Φ with a trilinear multiplication $\{abc\}$ and satisfying

$$\{abc\} = \{acb\}, \tag{1.1}$$

$$\begin{aligned}&\{ab\{cde\}\} + \{\{bac\}de\} + \{ce\{bad\}\} + \{cd\{bae\}\} \\ &\quad = \{ba\{cde\}\} + \{\{abc\}de\} + \{ce\{abd\}\} + \{cd\{abe\}\}.\end{aligned} \tag{1.2}$$

DEFINITION 1.2. A *J. t. s.* J_{II} *of type* II is a vector space over a field Φ with a

1) Numbers in brackets refer to the references at the end of the paper.

trilinear multiplication $<abc>$[2] and satisfying

$$<abc>=<cba>, \tag{1.3}$$

$$\begin{aligned} &<ab<cde>> + <<bac>de> + <c<bad>e> + <cd<bae>> \\ &= <ba<cde>> + <<abc>de> + <c<abd>e> + <cd<abe>>. \end{aligned} \tag{1.4}$$

EXAMPLES. A vector space over Φ spanned by the Dirac matrices $\gamma_i, i=1,2,3,4,5$, is a J.t.s. of type I relative to a ternary composition $\{\gamma_i\{\gamma_j\gamma_k\}\}$, since they satisfy the relation $\{\gamma_i\gamma_j\}=2\delta_{ij}I$, I being a unit matrix.

Consider a wave equation $\partial_\mu\beta_\mu\Psi+\kappa\Psi=0$ for the meson, then the operators β_i, $i=1,2,3,4$, satisfy, by definition, the following relation

$$\beta_i\beta_j\beta_k+\beta_k\beta_j\beta_i=\delta_{ij}\beta_k+\delta_{kj}\beta_i\text{[3]}.$$

Therefore, a vector space spanned by these β-matrices is a J.t.s. of type II relative to a ternary composition $<\beta_i\beta_j\beta_k>=\beta_i\beta_j\beta_k+\beta_k\beta_j\beta_i$. A J.t.s. of this type is called a meson triple system [7].

Let J be a subspace of an associative algebra A, which is closed with respect to a ternary composition $\{a\{bc\}\}$, then J is a J.t.s. of type I relative to $\{abc\}=\{a\{bc\}\}$ and it is a J.t.s. of type II relative to $<abc>=abc+cba$.

DEFINITION 1.3. A linear transformation D of J.t.s. J_{I} of type I is called a *derivation* of J_{I} if

$$D\{xyz\}=\{(Dx)yz\}+\{x(Dy)z\}+\{xy(Dz)\}$$

for all x,y,z in J_{I}. A derivation of a J.t.s. of type II is defined similarly.

In a J.t.s. J_{I} of type I, if we put

$$[abc]=\{abc\}-\{bac\},$$

then (1.2) can be rewritten as

$$[ab\{cde\}]=\{[abc]de\}+\{c[abd]e\}+\{cd[abe]\}, \tag{1.5}$$

hence a linear mapping $\sum_i D_{(a_i,b_i)}\colon x\to\sum_i[a_ib_ix]$ is a derivation of J_{I}, which is called an inner derivation of J_{I}.

Similarly, for a J.t.s. J_{II} of type II if we put

$$[abc]=<abc>-<bac>,$$

then (1.4) shows that a linear mapping $\sum_i D_{(a_i,b_i)}\colon x\to\sum_i[a_ib_ix]$ is an (inner) derivation of J_{II}, in each case the inner derivations form an ideal of a Lie algebra $\mathfrak{D}$ generated by derivations of J_{I} (or J_{II}), since $[D,D_{(a,b)}]=D_{(Da,b)}+D_{(a,Db)}$ for every $D\in\mathfrak{D}$.

2) In [7], Jacobson denoted a ternary product $abc+cba$ in associative algebras by (abc). Later he generalized this product to the case of abstract Jordan algebras and denoted by $\{abc\}$ [10]. Here we use the notation $<abc>$ instead of $\{abc\}$. See Lemma 1.1.

3) Duffin [4] and Kemmer [11].

Now, we consider the relations of J. t. s. with the other algebras. A (nonassociative) commutative algebra J over a field Φ is called a Jordan algebra if $(a^2b)a=a^2(ba)$ for all $a, b \in J$.

LEMMA 1.1. *Any Jordan algebra J is a J.t.s. of type I relative to $\{abc\}=a(bc)$ and a J.t.s. of type II relative to* $\langle abc\rangle=\frac{1}{2}a(bc)-\frac{1}{2}b(ca)+\frac{1}{2}c(ab)$.

Proof. In a Jordan algebra J, it holds that $a(b(cd))+c(b(ad))+d(b(ca))=a(d(bc))+b(d(ca))+c(d(ab))$ for all $a,b,c,d\in J$ [1, (7)]. Hence a linear mapping $x\to[abx]$ is a derivation of a Jordan algebra J, where $[abc]=a(bc)-b(ac)$. From this fact we can easily prove this lemma.

LEMMA 1.2. *A J.t.s. of type I is a J.t.s. of type II relative to* $\langle abc\rangle=\frac{1}{2}\{abc\}-\frac{1}{2}\{bca\}+\frac{1}{2}\{cab\}$. *Conversely a J.t.s. of type II is a J.t.s. of type I relative to* $\{abc\}=\langle abc\rangle+\langle acb\rangle$. In this case $J_{\rm II}$ (or $J_{\rm I}$) is called an *associated J.t.s.* of $J_{\rm I}$ (or $J_{\rm II}$).

A vector space T over Φ with a trilinear composition $[abc]$ is called a Lie triple system if

$$[aab]=0,$$
$$[abc]+[bca]+[cab]=0,$$
$$[ab[cde]]=[[abc]de]+[c[abd]e]+[cd[abe]].$$

The following lemma will be usefull in the sequel.

LEMMA 1.3. *A J.t.s. $J_{\rm I}$ of type I is a Lie triple system relative to $[abc]=\{abc\}-\{bac\}$ and a J.t.s. $J_{\rm II}$ of type II is a Lie triple system relative to* $[abc]=\langle abc\rangle-\langle bac\rangle$.

To prove this lemma we use that a linear mapping $x\to[abx]$ is a derivation of $J_{\rm I}$ (or $J_{\rm II}$). A corollary of this lemma is that any Jordan algebra is a Lie triple system relative to $[abc]=a(bc)-b(ac)$ [8]. We call a Lie triple system derived from $J_{\rm I}$ (or $J_{\rm II}$) by above lemma the *associated Lie triple system* of $J_{\rm I}$ (or $J_{\rm II}$).

We prove now an identity in Jordan algebras:

$$(1.6)\qquad [\langle aba\rangle, b, ab]=\frac{1}{2}\,a((ab)(b(ab)))-\frac{1}{2}\,b((ab)(a(ba)))-\frac{1}{4}\,[a^2, b^2, ab].$$

Proof. In a Jordan algebra the following identity holds [1, (6)]:

$$(1.7)\qquad a((bc)d)+b((ca)d)+c((ab)d)=(ab)(cd)+(bc)(ad)+(ca)(bd).$$

From this, we have (α) $2(a(ab))(b(ab))=a((ab)(b(ab)))+b((ab)(a(ba)))$. If we apply (1.7) with $a=b$, $b=a^2$, $c=b$, $d=ab$, we obtain $2((a^2b)(b(ab))-b((a^2b)(ab)))=a^2(b^2(ab))-b^2(a^2(ab))$, i.e. (β) $[a^2b,\ b,\ ab]=\frac{1}{2}\,[a^2, b^2, ab]$. Using (α) and (β) we obtain (1.6). If we use (1.6) and [10, (19)], we see that the expression $[a, b, \langle aba\rangle b]-[\langle aba\rangle, b, ab]$ is skew-symmetric in a and b.

REMARK 1.1. We consider the geometrical meanning of the axiom (1.2) or (1.5) for a J.t.s. J_{I} of type I. Let L_n be a space with a symmetric affine connection and suppose that the curvature tensor $R_{ijk}{}^l$ in L_n decomposes as $R_{ijk}{}^l = 2K_{[ij]k}{}^l$, where the tensor $K_{ijk}{}^l$ satisfies $K_{i[jk]}{}^l = 0$ and $\Gamma_m K_{ijk}{}^l = 0$, Γ denoting a covariant derivation. If we apply the operator $\Gamma_{[e}\Gamma_{f]}$ to $K_{ijk}{}^l$ we have

$$R_{efm}{}^l K_{ijk}{}^m - R_{eji}{}^m K_{mjk}{}^l - R_{efj}{}^m K_{imk}{}^l - R_{efk}{}^m K_{ijm}{}^l = 0.$$

This relation is an expression of (1.5) in the form of structure constants of J_{I} and the curvature tensor $R_{ijk}{}^l$ satisfies three conditions in coefficient forms for the associated Lie triple system of J_{I}. For the geometrical meanning of the axiom (1.4) for a J.t.s. of type II we can consider a space L_n whose curvature tensor satisfies a special identity $R_{ijk}{}^l = 2K_{[ij]k}{}^l$, where $K_{[i|j|k]}{}^l = 0$ and $\Gamma_m K_{ijk}{}^l = 0$.

REMARK 1.2. Let J be a J.t.s. of type I. Since a Lie triple system T can be 1-to-1 imbedded in a Lie algebra L in such a way that the given composition $[abc]$ in T is a product $[[ab]c]$ in L [8 § 5, 14 Theorem 2.1], we can consider J as a subspace of a Lie algebra such that $\{abc\} - \{bac\} = [[ab]c]$ by Lemma 1.3.

We shall now state some concepts for J.t.s. of type I[4] which will be necessary for the later use. Let J and J' be J.t.s. of type I over Φ, a homomorphism of J into J' is a linear mapping f of J into J' satisfying $f(\{abc\}) = \{f(a)f(b)f(c)\}$ for all $a,b,c \in J$. A subspace K of J is called a subsystem of J if $a, b, c \in K$ implies $\{abc\} \in K$. A subspace K of J is a subsystem if and only if $[abb] \in K$ and $\{aaa\} \in K$ for all $a, b \in K$. In fact, suppose that a subspace K satisfies these conditions, then $[abc] + [acb] \in K$ and $[bac] + [bca] \in K$, hence $3[abc] \in K$ and K is a subsystem of the associated Lie triple system of J. Also, using the identity $2(\{abc\} + \{bca\} + \{cab\}) = \{a+b+c, a+b+c, a+b+c\} - \{a+b, a+b, a+b\} - \{b+c, b+c, b+c\} - \{c+a, c+a, c+a\} + \{aaa\} + \{bbb\} + \{ccc\}$ we have $\{abc\} + \{bca\} + \{cab\} \in K$ hence $3\{abc\} = [abc] + [acb] + \{abc\} + \{bca\} + \{cab\} \in K$. An ideal of J is a subspace K satisfying $\{JJK\} \subseteqq K$ and $\{KJJ\} \subseteqq K$. Let K be an ideal of J, then the factor space J/K becomes a J.t.s. with a trilinear product $\{a+K\ b+K\ c+K\} = \{abc\} + K$ and a natural mapping $a \to a+K$ is a homomorphism of J onto J/K. Conversely, let f be a homomorphism of a J.t.s. J onto a J.t.s. J' with a kernel K, then K is an ideal of J and the factor system J/K is isomorphic to J'. An ideal K of a J.t.s. J is a Lie triple system ideal[5] of an associated Lie triple system of J, because $[kab] = \{kab\} - \{abk\}$, $a, b \in J$, $k \in K$ and this relation shows that an ideal K of an associated Lie triple system of a J.t.s. J is an ideal of J if and only if either $\{JJK\} \subseteqq K$ or $\{KJJ\} \subseteqq K$.

2. Representations of Jordan triple systems. In this section we define a general representation of J.t.s. and consider the relations among the representations of Jordan algebras, of J.t.s. and of Lie triple systems. We begin with the natural definition of

4) In [13] Ôno stated these concepts in the form $\{a\{bc\}\}$.
5) [12, Definition 1.3].

representations for J. t. s.

DEFINITION 2.1. A linear mapping ρ of a J. t. s. J_{I} of type I into the algebra of linear transformations on a vector space V over Φ is called a *special representation* of J_{I} if $\rho(\{abc\})=\{\rho(a)\{\rho(b)\rho(c)\}\}$.

But, for our purpose it is necessary to define a more generalized representation than the special representation.

DEFINITION 2.2. Let J_{I} be a J. t. s. of type I. A pair (L, R) of bilinear mappings of J_{I} into the algebra of linear transformations on a vector space V over Φ is called a *(bi-)representation* of J_{I} if

$$R(a, b)=R(b, a), \tag{2.1}$$

$$\begin{aligned} &R(a, \{bcd\})-L(a, \{bcd\}) \\ &\quad =R(c, d)(R(a, b)-L(a, b))+L(b, d)(R(a, c)-L(a, c)) \\ &\qquad +L(b, c)(R(a, d)-L(a, d)), \end{aligned} \tag{2.2}$$

$$[L(a, b)-L(b, a),\ R(c, d)]=R([abc], d)+R(c, [abd]), \tag{2.3}$$

$$[L(a, b)-L(b, a),\ L(c, d)]=L([abc], d)+L(c, [abd]), \tag{2.4}$$

where $[L(a, b), R(c, d)]$ denotes, as usual, $L(a, b)\,R(c, d)-R(c, d)\,L(a, b)$.

From (2.4) we have $[L(a, b)-L(b, a), L(c, d)-L(d, c)]=L([abc], d)-L(d, [abc])+L(c, [abd])-L([abd], c)$, hence $\sum_i (L(a_i, b_i)-L(b_i, a_i))$ generate a subalgebra $\mathfrak{L}$ of a Lie algebra $\mathfrak{gl}(V)$. Let K be an ideal of J_{I} and let $(\bar{L}, \bar{R})$ be a restriction of a representation (L, R) of J_{I} to K, then a Lie algebra generated by $\sum_i (\bar{L}(a_i, b_i)-\bar{L}(b_i, a_i)), a_i, b_i \in K$, is an ideal of $\mathfrak{L}$.

For a, b in J_{I}, if we denote the mapping $x \to \{abx\}$ by $L(a, b)$ and the mapping $x \to \{xab\}$ by $R(a, b)$, then L and R satisfy (2.1), $\cdots$, (2.4). As usual we call this representation (L, R) the *regular representation.* We note that if (L, R) is a regular representation, then $L(a, b)-L(b, a)$ is an inner derivation of J_{I}. If K is a subsystem of J_{I} and A is an ideal of J_{I}, then the regular representation (L, R) induces a representation of K in A. If k is in a kernel of the regular representation, then $[kab]=0$ for every a, b in J_{I}. The inner derivation $\sum_i (L(a_i, b_i)-L(b_i, a_i))$ becomes a trivial mapping on the set of all elements with this property.

Let ρ be a special representation of J_{I}, then we have $\rho([abc])=[[\rho(a)\rho(b)]\rho(c)]$. Hence, if we put $L(a, b)=\rho(a)\rho(b)$ and $R(a, b)=\rho(a)\rho(b)+\rho(b)\rho(a)$, then it follows that (L, R) is a representation of J_{I}.

DEFINITION 2.3. Let J_{II} be a J. t. s. of type II. A pair (λ, τ) of bilinear mappings of J_{II} into the algebra of linear transformations on a vector space V over Φ is called a *(bi-)representation* of J_{II} if

$$\tau(a, b)=\tau(b, a), \tag{2.5}$$

$$(2.6)\qquad \begin{aligned}&\tau(\langle abc\rangle, d)-\lambda(\langle abc\rangle, d)\\ &\quad=\tau(a,c)(\tau(b,d)-\lambda(b,d))+\lambda(a,b)(\tau(c,d)-\lambda(c,d))\\ &\qquad\qquad+\lambda(c,b)(\tau(a,d)-\lambda(a,d)),\end{aligned}$$

$$(2.7)\qquad [\lambda(a,b)-\lambda(b,a), \tau(c,d)]=\tau([abc],d)+\tau(c,[abd]),$$

$$(2.8)\qquad [\lambda(a,b)-\lambda(b,a), \lambda(c,d)]=\lambda([abc],d)+\lambda(c,[abd]).$$

For a, b in a J. t. s. J_{II} of type II, two linear mappings $\lambda(a,b)\colon x\to\langle abx\rangle$ and $\tau(a,b)\colon x\to\langle axb\rangle$ satisfy the conditions (2.5), $\cdots$, (2.8), hence (λ,τ) is a representation of J_{II} which we call a regular representation. In this case $\lambda(a,b)-\lambda(b,a)$ is an inner derivation of J_{II}.

LEMMA 2.1. *Let $a\to\rho(a)$ be a representation*[6] of a *Jordan algebra J. Then ρ induces a representation (L,R) for an associated J. t. s. of type I of J.*

In fact, if we put $L(a,b)=\rho(a)\rho(b)$ and $R(a,b)=\rho(ab)$, then we easily see that L and R satisfy the conditions (2.1), $\cdots$, (2.4). Also, from Lemma 1.2 we have the following lemma by a direct verification.

LEMMA 2.2. *Let (L,R) be a representation for a J. t. s. J_{I} of type I. If we put $\lambda(a,b)=\frac{1}{2}L(a,b)-\frac{1}{2}L(b,a)+\frac{1}{2}R(a,b)$ and $\tau(a,b)=\frac{1}{2}L(a,b)+\frac{1}{2}L(b,a)-\frac{1}{2}R(a,b)$, then (λ,τ) is a representation for an associated J. t. s. of type II of J_{I}. Conversely, let (λ,τ) be a representation for a J. t. s. J_{II} of type II. If we put $L(a,b)=\lambda(a,b)+\tau(a,b)$ and $R(a,b)=\lambda(a,b)+\lambda(b,a)$, then (L,R) is a representation for an associated J. t. s. of type I of J_{II}.*

LEMMA 2.3. *Let (L,R) be a representation for a J. t. s. of type I. If we put $\theta(a,b)=R(a,b)-L(a,b)$, then we have*

$$\theta(c,d)\theta(a,b)-\theta(b,d)\theta(a,c)-\theta(a,[bcd])+D(b,c)\theta(a,d)=0,$$
$$[D(a,b),\theta(c,d)]=\theta([abc],d)+\theta(c,[abd]),$$

where $D(a,b)=\theta(b,a)-\theta(a,b)$.

From Lemma 1.3 and Lemma 2.3, it follows that a representation (L,R) of a J. t. s. J_{I} of type I induces a representation θ[7] of an associated Lie triple system of J_{I}. Similarly, a representation (λ,τ) of a J. t. s. J_{II} of type II induces a representation θ of an associated Lie triple system of J_{II} by putting $\theta(a,b)=\lambda(b,a)-\tau(b,a)$. Therefore, by [15] from a representation of a J. t. s. J we can define a cohomology group of an associated Lie triple system of J.

For a Jordan algebra J, there is a concept of Jordan bimodule equivalent to that of representation for J. In the case of J. t. s. of type I we define a Jordan triple bimodule

6) [8, Definition 2.1].

7) See [15, Definition 2], there we called a representation space V a $\mathfrak{T}$-module instead of a representation θ.

as follows.

Let J_{I} be a J.t.s. of type I. A *Jordan triple bimodule* for J_{I} is a vector space M with trilinear compositions $\{mab\}$, $\{amb\}$ and $\{abm\}$ for $a, b\in J_{\mathrm{I}}$, $m\in M$ such that these compositions are contained in M and satisfy

$$\begin{aligned}
&\{mab\}=\{mba\},\\
&\{amb\}=\{abm\},\\
&\{ma\{bcd\}\}+\{\{abm\}cd\}+\{bd\{acm\}\}+\{bc\{adm\}\}\\
&\quad=\{am\{bcd\}\}+\{\{mab\}cd\}+\{bd\{mac\}\}+\{bc\{mad\}\},\\
&\{ab\{mcd\}\}+\{\{bam\}cd\}+\{md\{bac\}\}+\{mc\{bad\}\}\\
&\quad=\{ba\{mcd\}\}+\{\{abm\}cd\}+\{md\{abc\}\}+\{mc\{abd\}\},\\
&\{ab\{cdm\}\}+\{cd\{bam\}\}+\{\{bac\}dm\}+\{cm\{bad\}\}\\
&\quad=\{ba\{cdm\}\}+\{cd\{abm\}\}+\{\{abc\}dm\}+\{cm\{abd\}\}.
\end{aligned}$$

Let (L, R) be a representation of J_{I} acting in the vector space M. Putting $\{abm\}=\{amb\}=L(a, b)m$ and $\{mab\}=R(a, b)m$ for $m\in M$, M is a Jordan triple bimodule for J_{I}. Conversely, let M be a Jordan triple bimodule for J_{I}. If we define the linear mappings $L(a, b)$ and $R(a, b)$ of M by $L(a, b)m=\{abm\}$ and $R(a, b)m=\{mab\}$ respectively, then (L, R) is a representation of J_{I} with representation space M. Hence, the concept of Jordan triple bimodule for J_{I} is equivalent to that of representation of J_{I}. As in [9, § 2] from a given Jordan triple bimodule M we can construct a semi-direct sum of J_{I} and the bimodule M or a split null extension of J_{I} by M, and we can discuss this problem in a more general situation.

3. Cohomology group of Jordan triple systems of type I. Let (L, R) be a representation of a J.t.s. J_{I} of type I acting in a vector space V over Φ, and let f be an n-linear mapping of $J_{\mathrm{I}}\times\cdots\times J_{\mathrm{I}}$ (n times) into V satisfying

$$f(x_1, \cdots, x_{n-2}, x, y)=f(x_1, \cdots, x_{n-2}, y, x) \qquad \text{for } n\geqq 3.$$

We call such a mapping f an *n-cochain* and denote a vector space spanned by n-cochains by $C^n(J_{\mathrm{I}}, V)$, $n=0, 1, 2, \cdots$, where we identify $C^0(J_{\mathrm{I}}, V)$ with V.

We define a linear mapping δ of $C^n(J_{\mathrm{I}}, V)$ into $C^{n+2}(J_{\mathrm{I}}, V)$ as follows:

$$(\delta f)(x_1, x_2)=(L(x_1, x_2)-R(x_1, x_2))f \qquad \text{for } f\in C^0(J_{\mathrm{I}}, V), \tag{3.1}$$

$$\begin{aligned}(\delta f)(x_1, x_2, x_3)=&L(x_1, x_2)f(x_3)+L(x_1, x_3)f(x_2)\\&+R(x_2, x_3)f(x_1)-f(\{x_1x_2x_3\})\end{aligned} \qquad \text{for } f\in C^1(J_{\mathrm{I}}, V), \tag{3.2}$$

$$\begin{aligned}&(\delta f)(x_1, x_2, x_3, x_4)\\&\quad=L(x_2, x_3)f(x_1, x_4)+L(x_2, x_4)f(x_1, x_3)\\&\qquad+R(x_3, x_4)f(x_1, x_2)-f(x_1, \{x_2x_3x_4\})\end{aligned} \qquad \text{for } f\in C^2(J_{\mathrm{I}}, V), \tag{3.3}$$

$$
\begin{aligned}
&(\delta f)(x_1, x_2, \cdots, x_{2n+1}) \\
&= (-1)^{n+1}[L(x_{2n-1}, x_{2n})f(x_1, x_2, \cdots, x_{2n-2}, x_{2n+1}) \\
&\qquad\qquad -L(x_{2n-1}, x_{2n})f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n+1}) \\
&\quad +L(x_{2n-1}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-2}, x_{2n}) \\
&\qquad\qquad -L(x_{2n-1}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n}) \qquad (3.4)\\
&\quad +R(x_{2n}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-1}) - R(x_{2n}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n-1}) \\
&\quad -f(x_1, x_2, \cdots, x_{2n-2}, \{x_{2n-1}x_{2n}x_{2n+1}\}) + f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, \{x_{2n-1}x_{2n}x_{2n+1}\})] \\
&\quad +\sum_{k=1}^{n-1}(-1)^{k+1}(L(x_{2k-1}, x_{2k}) - L(x_{2k}, x_{2k-1}))f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1}) \\
&\quad +\sum_{k=1}^{n-1}\sum_{j=2k+1}^{2n+1}(-1)^k f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2n+1}) \\
&\qquad\qquad\qquad\qquad \text{for} \quad f \in C^{2n-1}(J_{\mathrm{I}}, V),\ n = 2, 3, \cdots,
\end{aligned}
$$

$$
\begin{aligned}
&(\delta f)(y, x_1, x_2, \cdots, x_{2n+1}) \\
&= (-1)^{n+1}[L(x_{2n-1}, x_{2n})f(y, x_1, x_2, \cdots, x_{2n-2}, x_{2n+1}) \\
&\qquad\qquad -L(x_{2n-1}, x_{2n})f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n+1}) \\
&\quad +L(x_{2n-1}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-2}, x_{2n}) \\
&\qquad\qquad -L(x_{2n-1}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n}) \\
&\quad +R(x_{2n}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-1}) \qquad (3.5)\\
&\qquad\qquad -R(x_{2n}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n-1}) \\
&\quad -f(y, x_1, x_2, \cdots, x_{2n-2}, \{x_{2n-1}x_{2n}x_{2n+1}\}) \\
&\qquad\qquad +f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, \{x_{2n-1}x_{2n}x_{2n+1}\})] \\
&\quad +\sum_{k=1}^{n-1}(-1)^{k+1}(L(x_{2k-1}, x_{2k}) - L(x_{2k}, x_{2k-1}))f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1}) \\
&\quad +\sum_{k=1}^{n-1}\sum_{j=2k+1}^{2n+1}(-1)^k f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2n+1}) \\
&\qquad\qquad\qquad\qquad \text{for} \quad f \in C^{2n}(J_{\mathrm{I}}, V),\ n = 2, 3, \cdots,
\end{aligned}
$$

where the sign ^ over a letter indicates that this letter is to be omitted.

Then, we obtain the following

THEOREM 3.1. *For the operator δ defined above, we have $\delta\delta f = 0$ for every cochain f.*

Proof. If $f \in C^0(J_{\mathrm{I}}, V)$, then

$$
\begin{aligned}
&(\delta\delta f)(x_1, x_2, x_3, x_4) \\
&= (R(x_1, \{x_2x_3x_4\}) - L(x_1, \{x_2x_3x_4\}) + R(x_3, x_4)(L(x_1, x_2) - R(x_1, x_2)) \\
&\qquad + L(x_2, x_4)(L(x_1, x_3) - R(x_1, x_3)) + L(x_2, x_3)(L(x_1, x_4) - R(x_1, x_4)))f = 0
\end{aligned}
$$

by (2.2). Similarly, a direct verification shows that $\delta\delta f = 0$ for 1- and 3-cochain f.

To prove the general case, it is useful to consider two linear mappings. For a pair a, b in J_{I} we define a linear mapping $\kappa(a, b)$ of $C^{2n-1}(J_{\mathrm{I}}, V)$ into $C^{2n-1}(J_{\mathrm{I}}, V)$ and a linear mapping $\iota(a, b)$ of $C^{2n-1}(J_{\mathrm{I}}, V)$ into $C^{2n-3}(J_{\mathrm{I}}, V)$ by the following formulas:

$$(3.6)\qquad \begin{aligned}(\kappa(a,b)f)(x_1,\cdots,x_{2n-1})&=(L(a,b)-L(b,a))f(x_1,\cdots,x_{2n-1})\\&\quad-\sum_{j=1}^{2n-1}f(x_1,\cdots,[abx_j],\cdots,x_{2n-1}),\qquad n=2,3,4,\cdots,\end{aligned}$$

$$(3.7)\qquad (\iota(a,b)f)(x_1,\cdots,x_{2n-3})=f(a,b,x_1,\cdots,x_{2n-3}),\qquad n=3,4,5,\cdots.$$

Then, we have by a direct calculation the following two formulas for $f\in C^{2n-1}(J_{\mathrm{I}},V), n\geqq 3$,

$$(3.8)\qquad \iota(a,b)\delta f+\delta\iota(a,b)f=\kappa(a,b)f,$$

$$(3.9)\qquad [\kappa(a,b),\iota(c,d)]f=\iota([abc],d)f+\iota(c,[abd])f.$$

Next we have

$$(3.10)\qquad [\kappa(a,b),\kappa(c,d)]f=\kappa([abc],d)f+\kappa(c,[abd])f \qquad \text{for}\quad f\in C^{2n-1}(J_{\mathrm{I}},V),\ n\geqq 3.$$

For if $f\in C^5(J_{\mathrm{I}},V)$, then we obtain (3.10) directly. Using Lemma 1.3 and (3.9) we can prove the general case by an inductive method. Moreover, we have

$$(3.11)\qquad \kappa(a,b)\delta f=\delta\kappa(a,b)f \qquad \text{for}\quad f\in C^{2n-1}(J_{\mathrm{I}},V),\ n\geqq 3.$$

For if $f\in C^5(J_{\mathrm{I}},V)$, then we obtain (3.11) by a direct computation. The general case follows by using the induction and (3.8). We have next for all $f\in C^n(J_{\mathrm{I}},V)$

$$(3.12)\qquad \delta\delta f=0.$$

Since (3.12) holds in case of $n=0,1,3$, we assume that (3.12) has been proved for all $f\in C^{2n-3}(J_{\mathrm{I}},V)$ and suppose $f\in C^{2n-1}(J_{\mathrm{I}},V), n\geqq 3$. Then for every pair $a,b\in J_{\mathrm{I}}$, from (3.8) and (3.11)

$$\begin{aligned}\iota(a,b)\delta\delta f&=\kappa(a,b)\delta f-\delta\iota(a,b)\delta f\\&=\delta\delta\iota(a,b)f\\&=0.\end{aligned}$$

Hence (3.12) holds for all cochains f with odd dimension and from this by (3.3) and (3.5), (3.12) holds for all cochains f with even dimension. Theorem 3.1 is therefore proved.

An n-cochain f is called an *n-cocycle* if $\delta f=0$. Denote $Z^n(J_{\mathrm{I}},V)$ a subspace of $C^n(J_{\mathrm{I}},V)$ spanned by n-cocycles. An n-cochain f of the form δg, where $g\in C^{n-2}(J_{\mathrm{I}},V)$, is called an *n-coboundary*. We denote $B^n(J_{\mathrm{I}},V)$ a subspace of $C^n(J_{\mathrm{I}},V)$ spanned by n-coboundaries, where $B^0(J_{\mathrm{I}},V)=B^1(J_{\mathrm{I}},V)=0$ by definition. Then, by Theorem 3.1 $B^n(J_{\mathrm{I}},V)$ is a subspace of $Z^n(J_{\mathrm{I}},V)$, hence we can define the quotient space $H^n(J_{\mathrm{I}},V)=Z^n(J_{\mathrm{I}},V)/B^n(J_{\mathrm{I}},V)$ which is called the *nth cohomology group* of J_{I}.

From Lemma 2.3 a representation (L,R) of J_{I} with representation space V induces a representation θ of an associated Lie triple system of J_{I}. Hence

$H^0(J_{\mathrm{I}}, V)$ is the subspace of the invariant elements for the induced representation θ of V.

A linear mapping f of J_{I} into V is called a derivation of J_{I} into V if $f(\{x_1x_2x_3\}) = R(x_2, x_3)f(x_1) + L(x_1, x_3)f(x_2) + L(x_1, x_2)f(x_3)$. Then,

$H^1(J_{\mathrm{I}}, V)$ is the vector space spanned by derivations of J_{I} into V.

We shall next consider the relation between a cohomology group $H^n(J_{\mathrm{I}}, V)$ of a J. t. s. J_{I} of type I and a cohomology group $H^n(J_{\mathrm{I}}^*, V)$ of an associated Lie triple system J_{I}^* of J_{I}. For this purpose, we modify slightly the coboundary operator δ introduced for Lie triple system in [15, (10), (11), (12)] as follows. Thus, for a Lie triple system J_{I}^*, let $C^n(J_{\mathrm{I}}^*, V)$ be a vector space spanned by n-cochains. We define a linear mapping $f \to \delta^* f$ of $C^n(J_{\mathrm{I}}^*, V)$ into $C^{n+2}(J_{\mathrm{I}}^*, V)$ by the following formulas:

$$(\delta^* f)(x_1, x_2) = -(\delta f)(x_1, x_2) \qquad \text{for } f \in C^0(J_{\mathrm{I}}^*, V),$$

$$(\delta^* f)(x_1, x_2, \cdots, x_{2n+1}) = (-1)^{n+1}(\delta f)(x_1, x_2, \cdots, x_{2n+1}) \qquad \text{for } f \in C^{2n-1}(J_{\mathrm{I}}^*, V),\ n \geqq 1,$$

$$(\delta^* f)(y, x_1, x_2, \cdots, x_{2n+1}) = (-1)^{n+1}(\delta f)(y, x_1, x_2, \cdots, x_{2n+1}) \qquad \text{for } f \in C^{2n}(J_{\mathrm{I}}^*, V),\ n \geqq 1.$$

Then $\delta^*\delta^* f = 0$ for all $f \in C^n(J_{\mathrm{I}}^*, V)$ by [15, Theorem 1], and we define the nth cohomology group $H^n(J_{\mathrm{I}}^*, V)$ of J_{I}^* as the quotient space $Z^n(J_{\mathrm{I}}^*, V)/B^n(J_{\mathrm{I}}^*, V)$.

Let $f \in C^n(J_{\mathrm{I}}, V)$. Define an n-linear mapping g of $J_{\mathrm{I}}^* \times \cdots \times J_{\mathrm{I}}^*$ (n times) into V as

$$g = f \qquad \text{for } f \in C^n(J_{\mathrm{I}}, V),\ n = 0, 1, 2,$$

$$g(x_1, \cdots, x_n) = f(x_1, \cdots, x_n) - f(x_1, \cdots, x_{n-3}, x_{n-1}, x_{n-2}, x_n) \qquad \text{for } f \in C^n(J_{\mathrm{I}}, V),\ n \geqq 3,$$

then the mapping ϕ: $f \to g$ is a linear mapping of $C^n(J_{\mathrm{I}}, V)$ into $C^n(J_{\mathrm{I}}^*, V)$. Denote δ the coboundary operator for the elements of $C^n(J_{\mathrm{I}}, V)$ and δ^* the coboundary operator for the elements of $C^n(J_{\mathrm{I}}^*, V)$, then we have the following relations.

$$(\delta^* g)(x_1, x_2) = (\delta f)(x_1, x_2) \qquad \text{for } f \in C^0(J_{\mathrm{I}}, V),$$

$$(\delta^* g)(x_1, \cdots, x_{n+2}) = (\delta f)(x_1, \cdots, x_{n+2}) - (\delta f)(x_1, \cdots, x_{n-1}, x_{n+1}, x_n, x_{n+2}) \qquad \text{for } f \in C^n(J_{\mathrm{I}}, V),\ n \geqq 1.$$

Hence, $\phi(Z^n(J_{\mathrm{I}}, V)) \subseteq Z^n(J_{\mathrm{I}}^*, V)$ and $\phi(B^n(J_{\mathrm{I}}, V)) \subseteq B^n(J_{\mathrm{I}}^*, V)$ and from this ϕ induces a homomorphism ϕ^* of $H^n(J_{\mathrm{I}}, V)$ into $H^n(J_{\mathrm{I}}^*, V)$. Thus we obtain the following theorem.

THEOREM 3.2. *Let $H^n(J_{\mathrm{I}}, V)$ be the nth cohomology group of a J.t.s. J_{I} of type I and let $H^n(J_{\mathrm{I}}^*, V)$ be the nth cohomology group of an associated Lie triple system J_{I}^* of J_{I}. Then, there exists a homomorphism of $H^n(J_{\mathrm{I}}, V)$ into $H^n(J_{\mathrm{I}}^*, V)$.*

4. Cohomology group of Jordan triple systems of type II. Let J_{II} be a J. t. s. of type II and let (λ, τ) be a representation of J_{II} acting in a vector space V over Φ. An

n-cochain is an n-linear mapping f of $J_{\mathrm{II}}\times\cdots\times J_{\mathrm{II}}$ (n times) into V such that

$$f(x_1, \cdots, x_{n-3}, x, y, z)=f(x_1, \cdots, x_{n-3}, z, y, x) \qquad \text{for } n\geqq 3.$$

Denote $C^n(J_{\mathrm{II}}, V)$ a vector space spanned by n-cochains, where we define $C^0(J_{\mathrm{II}}, V)=V$. The coboundary operator is a linear mapping δ of $C^n(J_{\mathrm{II}}, V)$ into $C^{n+2}(J_{\mathrm{II}}, V)$ defined by the formulas:

$$(\delta f)(x_1, x_2)=(\tau(x_2, x_1)-\lambda(x_2, x_1))f \qquad \text{for } f\in C^0(J_{\mathrm{II}}, V), \tag{4.1}$$

$$\begin{aligned}(\delta f)(x_1, x_2, x_3)=\lambda(x_3, x_2)f(x_1)+\tau(x_1, x_3)f(x_2)&\\ +\lambda(x_1, x_2)f(x_3)-f(<x_1x_2x_3>)&\\ \text{for } f\in C^1(J_{\mathrm{II}}, V),&\end{aligned} \tag{4.2}$$

$$\begin{aligned}&(\delta f)(x_1, x_2, x_3, x_4)\\ &=\lambda(x_4, x_3)f(x_1, x_2)+\tau(x_2, x_4)f(x_1, x_3)\\ &\qquad+\lambda(x_2, x_3)f(x_1, x_4)-f(x_1, <x_2x_3x_4>)\\ &\qquad\qquad \text{for } f\in C^2(J_{\mathrm{II}}, V),\end{aligned} \tag{4.3}$$

$$\begin{aligned}&(\delta f)(x_1, x_2, \cdots, x_{2n+1})\\ &=(-1)^{n+1}[\lambda(x_{2n-1}, x_{2n})f(x_1, x_2, \cdots, x_{2n-2}, x_{2n+1})\\ &\qquad-\lambda(x_{2n-1}, x_{2n})f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n+1})\\ &\quad+\tau(x_{2n-1}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-2}, x_{2n})\\ &\qquad-\tau(x_{2n-1}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n})\\ &\quad+\lambda(x_{2n+1}, x_{2n})f(x_1, x_2, \cdots, x_{2n-1})\\ &\qquad-\lambda(x_{2n+1}, x_{2n})f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n-1})\\ &\quad-f(x_1, x_2, \cdots, x_{2n-2}, <x_{2n-1}x_{2n}x_{2n+1}>)\\ &\qquad+f(x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, <x_{2n-1}x_{2n}x_{2n+1}>)]\\ &\quad+\sum_{k=1}^{n-1}(-1)^{k+1}(\lambda(x_{2k-1}, x_{2k})-\lambda(x_{2k}, x_{2k-1}))f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1})\\ &\quad+\sum_{k=1}^{n-1}\sum_{j=2k+1}^{2n+1}(-1)^k f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2n+1})\\ &\qquad\qquad \text{for } f\in C^{2n-1}(J_{\mathrm{II}}, V), n=2, 3, \cdots,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&(\delta f)(y, x_1, x_2, \cdots, x_{2n+1})\\ &=(-1)^{n+1}[\lambda(x_{2n-1}, x_{2n})f(y, x_1, x_2, \cdots, x_{2n-2}, x_{2n+1})\\ &\qquad-\lambda(x_{2n-1}, x_{2n})f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n+1})\\ &\quad+\tau(x_{2n-1}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-2}, x_{2n})\\ &\qquad-\tau(x_{2n-1}, x_{2n+1})f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n})\\ &\quad+\lambda(x_{2n+1}, x_{2n})f(y, x_1, x_2, \cdots, x_{2n-1})\\ &\qquad-\lambda(x_{2n+1}, x_{2n})f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, x_{2n-1})\end{aligned} \tag{4.5}$$

$$
\begin{aligned}
&-f(y, x_1, x_2, \cdots, x_{2n-2}, \langle x_{2n-1}x_{2n}x_{2n+1}\rangle) \\
&\qquad\qquad +f(y, x_1, x_2, \cdots, x_{2n-4}, x_{2n-2}, x_{2n-3}, \langle x_{2n-1}x_{2n}x_{2n+1}\rangle)] \\
&+\sum_{k=1}^{n-1}(-1)^{k+1}(\lambda(x_{2k-1}, x_{2k})-\lambda(x_{2k}, x_{2k-1}))f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1}) \\
&+\sum_{k=1}^{n-1}\sum_{j=2k+1}^{2n+1}(-1)^k f(y, x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}x_{2k}x_j], \cdots, x_{2n+1})
\end{aligned}
$$

$$
\text{for } f\in C^{2n}(J_{\mathrm{II}}, V), n=2, 3, \cdots.
$$

where the sign ^ over a letter indicates that this letter is to be omitted.

Then, using the same method as in §3 we obtain

THEOREM 4.1. *For the operator δ defined above, we have $\delta\delta f=0$ for every cochain f.*

An n-cochain f with $\delta f=0$ is called an n-cocycle and a subspace of $C^n(J_{\mathrm{II}}, V)$ spanned by n-cocycles is denoted by $Z^n(J_{\mathrm{II}}, V)$. A cochain of the form δf, where $f\in C^{n-2}(J_{\mathrm{II}}, V)$ is called an n-coboundary and a subspace of $C^n(J_{\mathrm{II}}, V)$ spanned by n-coboundaries is denoted by $B^n(J_{\mathrm{II}}, V)$. By Theorem 4.1, $B^n(J_{\mathrm{II}}, V)$ is a subspace of $Z^n(J_{\mathrm{II}}, V)$. The quotient space $H^n(J_{\mathrm{II}}, V)=Z^n(J_{\mathrm{II}}, V)/B^n(J_{\mathrm{II}}, V)$ is called the nth *cohomology group* of J_{II}.

Let (L, R) be a representation of a J.t.s. J_{I} of type I with representation space V and let $H^n(J_{\mathrm{I}}, V)$ be the nth cohomology group of J_{I}. Then a representation (λ, τ) for an associated J.t.s. J_{II} of type II of J_{I} is induced from (L, R) and we have an n-th cohomology group $H^n(J_{\mathrm{II}}, V)$. Define a linear mapping $f\rightarrow\phi f$ of $C^n(J_{\mathrm{I}}, V)$ into $C^n(J_{\mathrm{II}}, V)$ by

$$
\phi f=f \qquad\qquad \text{for } f\in C^n(J_{\mathrm{I}}, V), n=0, 1, 2,
$$

$$
\begin{aligned}
(\phi f)(x_1, \cdots, x_n)=\frac{1}{2}f(x_1, \cdots, x_n)&-\frac{1}{2}f(x_1, \cdots, x_{n-3}, x_{n-1}, x_n, x_{n-2}) \\
&+\frac{1}{2}f(x_1, \cdots, x_{n-3}, x_n, x_{n-2}, x_{n-1})
\end{aligned}
$$

$$
\text{for } f\in C^n(J_{\mathrm{I}}, V), n\geqq 3,
$$

and define a linear mapping $g\rightarrow\varphi g$ of $C^n(J_{\mathrm{II}}, V)$ into $C^n(J_{\mathrm{I}}, V)$ by

$$
\varphi g=g \qquad\qquad \text{for } g\in C^n(J_{\mathrm{II}}, V), n=0, 1, 2,
$$

$$
(\varphi g)(x_1, \cdots, x_n)=g(x_1, \cdots, x_n)+g(x_1, \cdots, x_{n-2}, x_n, x_{n-1})
$$

$$
\text{for } g\in C^n(J_{\mathrm{II}}, V), n\geqq 3.
$$

Then, ϕ and φ are inverse isomorphisms since both $\varphi\phi$ and $\phi\varphi$ are identity mappings and $C^n(J_{\mathrm{I}}, V)\approx C^n(J_{\mathrm{II}}, V)$.

Denote $\delta_{\mathrm{I}} f$ a coboundary of $f\in C^n(J_{\mathrm{I}}, V)$ and denote $\delta_{\mathrm{II}} g$ a coboundary of $g\in C^n(J_{\mathrm{II}}, V)$, then we have by a direct calculation the following relations:

$$
(\delta_{\mathrm{II}}\phi f)(x_1, x_2)=(\delta_{\mathrm{I}} f)(x_1, x_2) \qquad\qquad \text{for } f\in C^0(J_{\mathrm{I}}, V),
$$

$$(\delta_{\mathrm{II}}\phi f)(x_1, \cdots, x_{n+2})$$
$$= \frac{1}{2}(\delta_{\mathrm{I}} f)(x_1, \cdots, x_{n+2}) - \frac{1}{2}(\delta_{\mathrm{I}} f)(x_1, \cdots, x_{n-1}, x_{n+1}, x_{n+2}, x_n)$$
$$+ \frac{1}{2}(\delta_{\mathrm{I}} f)(x_1, \cdots, x_{n-1}, x_{n+2}, x_n, x_{n+1})$$

for $f \in C^n(J_{\mathrm{I}}, V), n \geqq 1.$

Conversely,

$$(\delta_{\mathrm{I}}\varphi g)(x_1, x_2) = (\delta_{\mathrm{II}} g)(x_1, x_2) \qquad \text{for } g \in C^0(J_{\mathrm{II}}, V),$$

$$(\delta_{\mathrm{I}}\varphi g)(x_1, \cdots, x_{n+2}) = (\delta_{\mathrm{II}} g)(x_1, \cdots, x_{n+2}) + (\delta_{\mathrm{II}} g)(x_1, \cdots, x_n, x_{n+2}, x_{n+1})$$

for $g \in C^n(J_{\mathrm{II}}, V), n \geqq 1,$

From these relations ϕ maps $Z^n(J_{\mathrm{I}}, V)$ onto $Z^n(J_{\mathrm{II}}, V)$ and $Z^n(J_{\mathrm{I}}, V) \approx Z^n(J_{\mathrm{II}}, V)$. Assume that $f \in B^n(J_{\mathrm{I}}, V), n \geqq 2$, then f is the form $\delta_{\mathrm{I}} f'$ with $f' \in C^{n-2}(J_{\mathrm{I}}, V)$. We have $\phi f = \delta_{\mathrm{II}}\phi f'$ and ϕ maps $B^n(J_{\mathrm{I}}, V)$ onto $B^n(J_{\mathrm{II}}, V)$, therefore $B^n(J_{\mathrm{I}}, V) \approx B^n(J_{\mathrm{II}}, V), n \geqq 0.$

Thus we have the following theorem.

THEOREM 4.2. *Let $H^n(J_{\mathrm{I}}, V)$ be the nth cohomology group of a J.t.s. J_{I} of type I and let $H^n(J_{\mathrm{II}}, V)$ be the nth cohomology group of an associated J.t.s. J_{II} of type II of J_{I}. Then $H^n(J_{\mathrm{II}}, V)$ is isomorphic to $H^n(J_{\mathrm{I}}, V)$.*

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A NOTE ON REPRESENTATIONS OF ALGEBRAS AS SUBALGEBRAS OF $C(X)$ FOR X COMPACT

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(Received May 15, 1962)

In a recent paper Frank W. Anderson and Robert L. Blair gave some characterizations of $C(X)$ for X an arbitrary completely regular space [1][(1)].

In the paper, the theorem 4.1 states that *if A is a regular algebra, the condition $\mathfrak{M}_A=\mathfrak{R}_A$ is sufficient that A is isomorphic to a regular point-determining subalgebra of $C(X)$ for some topologically unique compact space X.*

But the converse is left open. We shall give in this note an example answering the question negatively.

§1. Some preliminary notions and notations.

We adopt the same notions and notations as in [1], but recite here some of them.

Let A be a subset of $C(X)$, A *is regular* in case (i) A contains the identity e of $C(X)$ and (ii) whenever $x\in X$ and U is an open neighborhood of x, there is an $f\in A$ such that $f(x)=0$ and $f(y)=1$ for all $y\in U$.

If A is any ring, then an ideal I in A is said to be *real* in case A/I is isomorphic to R real.

We shall denote by $\mathfrak{M}_A$, $\mathfrak{R}_A$ the set of all maximal ideals of A, and the set of all real maximal ideals of A respectively.

If X is a topological space and A is a subring of $C(X)$ then we shall set

$$M_x=\{f\in A;\ f(x)=0\}.$$

We say that A is point-determining in case $M\in\mathfrak{M}_A$ if and only if $M=M_x$ for some unique $x\in X$.

§2. A counter example to the converse of the theorem 4.1.

We define a compact set X and algebras A_0, A_1, A as follows.

X : $[0,1]$

A_0: the algebra of all functions $f\in C[0,1]$ such that for finite partition $0=a_0<a_1<\cdots<a_n=1$, $f(x)=p_i(x)/q_i(x)$ for $x\in[a_i, a_{i+1}]$, where p_i and q_i are polynomials. (The partition depends on f, and $q_i(x)\neq0$ for $x\in[a_i, a_{i+1}]$, $p_i(a_i)/q_i(a_i)=p_{i+1}(a_{i+1})/q_{i+1}(a_{i+1})$.)

(1)[1] Frank W. Anderson and Robert L. Blair: Characterizations of the algebra of all real-valued continuous functions on a completely regular space. Illinois Jour. of Math. vol. 3 (1959) pp. 121–133.

$\boldsymbol{A}_1$: the algebra generated by $\boldsymbol{A}_0$ and e^x.

$\boldsymbol{A}$: the algebra of all functions $f(x)$ of the form $g(x)/P(e^x)$, where g is an element of $\boldsymbol{A}_1$ and $P(e^x)$ is a function of the form $\prod_{k=1}^{n}(e^x-\alpha_k)$, $\alpha_k \in \boldsymbol{A}_0$, $e^x-\alpha_k \neq 0$ on $[0, 1]$.

Under these preparations we have following propositions.

(1) $\boldsymbol{A}_0$ *is a regular subset of* $C(\boldsymbol{X})$.

Proof. For any $X_1 \in \boldsymbol{X}$ and any $\varepsilon > 0$, the function defined by

$$f(x) = \begin{cases} 1 & \text{for } |x-x_1| \geqq \varepsilon \\ |x-x_1|/\varepsilon & \text{for } |x-x_1| \leqq \varepsilon, \end{cases}$$

is evidently contained in $\boldsymbol{A}_0$.

(2) $\boldsymbol{A}$ *is a regular subset of* $C(X)$.

This is evident from (1).

(3) *Let* $\boldsymbol{I}$ *be a real ideal of* $\boldsymbol{A}$. *Then* $\boldsymbol{I}_1 = \boldsymbol{A}_1 \cap \boldsymbol{I}$ *and* $\boldsymbol{I}_0 = \boldsymbol{A}_0 \cap \boldsymbol{I}$ *are real ideals.*

Proof. $\boldsymbol{A}_0/\boldsymbol{I}_0 \subset \boldsymbol{A}_1/\boldsymbol{I}_1 \subset \boldsymbol{A}/\boldsymbol{I}$. On the other hand, $1 \in \boldsymbol{A}_0$ and $\notin \boldsymbol{I}$. Since $\boldsymbol{A}_0$ is an algebra, $\boldsymbol{R} \subset \boldsymbol{A}_0/\boldsymbol{I}_0$. Hence $\boldsymbol{R} \subset \boldsymbol{A}_0/\boldsymbol{I}_0 \subset \boldsymbol{A}_1/\boldsymbol{I}_1 \subset \boldsymbol{A}/\boldsymbol{I} = \boldsymbol{R}$, which implies that $\boldsymbol{R} = \boldsymbol{A}_0/\boldsymbol{I}_0 = \boldsymbol{A}_1/\boldsymbol{I}_1$.

(4) *For the above ideal* $\boldsymbol{I}_0$, *there exists a point* $x_0 \in \boldsymbol{X}$ *such that* $\boldsymbol{I}_0 = \{f \in \boldsymbol{A};\ f(x_0) = 0\}$.

Proof. Let $g \in \boldsymbol{A}_0$. Then g has no zero point in $\boldsymbol{X}$ if and only if $1/g \in \boldsymbol{A}_0$. Hence if $g \in \boldsymbol{I}_0$, then g has at least one zero point. If $g, h \in \boldsymbol{I}_0$ such that g and h have no common zero point, then $g^2 + h^2 \in \boldsymbol{I}_0$ which has no zero point. This is a contradiction. Since X is a compact set, the ideal $\boldsymbol{I}_0$ has a common zero point. Since $\boldsymbol{I}_0$ is a real ideal, the common zero point are at most one.

(5) *The ideal* $\boldsymbol{I}_1$ *contains* $e^x - \alpha$ *for some* α, *where* $\log\alpha \in \boldsymbol{X}$.

Proof. $a_n u^n + a_{n-1}u^{n-1} + \cdots + a_0 + b \cdot Q(u) \in \boldsymbol{I}_1$, where $u = e^x$, $a_i \in \boldsymbol{A}_0$ and $a_i \notin \boldsymbol{I}_0$, $b \in \boldsymbol{I}_0$, $Q(u) \in \boldsymbol{A}_1$.

Since $\boldsymbol{I}_1 \supsetneqq \boldsymbol{A}_1\boldsymbol{I}_0$, we may assume $n \geqq 1$.

$a_n u^n + a_{n-1}u^{n-1} + \cdots + a_0 + b \cdot Q(u) \equiv a_n(x_0)u^n + a_{n-1}(x_0)u^{n-1} + \cdots + a_0(x_0) \pmod{\boldsymbol{A}_1\boldsymbol{I}_0}$. Hence $a_n(x_0)u^n + a_{n-1}(x_0)u^{n-1} + \cdots + a_0(x_0) \in \boldsymbol{I}_1$, $a_n(x_0) \neq 0$.

If $F(X) = a_n(x_0)X^n + a_{n-1}(x_0)X^{n-1} + \cdots + a_0(x_0)$ is irreducible, and if $n > 1$, then a root θ of $F(X) = 0$, which is not real, is contained in $\boldsymbol{A}_1/\boldsymbol{I}_1$. This is a contradiction. Hence $n = 1$, that is, $a_1(x_0)u + a_0(x_0) \in \boldsymbol{I}_1$. Since $a_1(x_0) \neq 0$, $\boldsymbol{I} \in u + a_0(x_0)/a_1(x_0) = u - \alpha$, where $\alpha = -a_0(x_0)/a_1(x_0)$.

If $\log\alpha \notin \boldsymbol{X}$, then $u - \alpha$ is a unit of $\boldsymbol{A}$, so $u - \alpha \notin \boldsymbol{I}$. Therefore $u - \alpha \in \boldsymbol{I}_1 \subset \boldsymbol{I}$ implies $\log\alpha \in \boldsymbol{X}$.

(6) $\log\alpha = x_0$. *That is,* $\boldsymbol{I}_1 = \{f \in \boldsymbol{A}_1;\ f(x_0) = 0\}$.

Proof. Suppose $\log\alpha \neq x_0$, for example, $x_0 > \log\alpha$. Then there exists $g \in \boldsymbol{I}_0$ such that $g \geqq 0$, and $g(x) > \alpha$ for $x \leqq \log\alpha$. $\boldsymbol{I}_1 \ni e^x - \alpha + g > 0$. Hence a unit $e^x - \alpha + g$ of $\boldsymbol{A}$ is contained in $\boldsymbol{I}$, which is a contradiction.

(7) *The algebra* $\boldsymbol{A}$ *is point-determining.*

Proof. For a real ideal $\boldsymbol{I}$, $\boldsymbol{I}_1=\boldsymbol{I}\cap\boldsymbol{A}_1$ determines a point $x_0\in\boldsymbol{X}$, Hence $\boldsymbol{I}=\boldsymbol{A}\boldsymbol{I}_1$ determines the point x_0.

(8) *The maximal ideal of* $\boldsymbol{A}$ *containing* $e^{2x}+1$ *is not real, that is* $\mathfrak{M}_A\neq\mathfrak{R}_A$.

Therefore we have the conclusion:

If $\boldsymbol{A}$ *is a regular algebra,* $\mathfrak{M}_A=\mathfrak{R}_A$ *is the sufficient but not the necessary condition that* $\boldsymbol{A}$ *is isomorphic to a regular point-determining subalgebra of* $C(\boldsymbol{X})$ *for some topologically unique compact space* $\boldsymbol{X}$.

Quite similarlly, the converse of the theorem 4.5 in [1] does not hold.

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GREEN'S FUNCTIONS FOR DIFFERENCE OPERATORS AND SOLUTIONS OF DIFFERENCE EQUATIONS

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(Received April 9, 1962)

§ 1. Introduction

When a linear homogeneous differential equation

$$Lu(x)=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

with some linear homogeneous boundary conditions is given, we often consider the *Green's function* of the operator L.

The Green's function $g(x,t)$ of the operator L with linear homogeneous boundary conditions

$$B_1(u)=0,\ B_2(u)=0,\ B_3(u)=0,\cdots\cdots, \quad \cdots\cdots\cdots\cdots (2)$$

is the solution of

$$Lg(x,t)=\delta(x-t) \quad \cdots\cdots\cdots\cdots\cdots\cdots (3)$$

satisfying the conditions $B_1(u)=0, B_2(u)=0,\cdots\cdots$, where $\delta(x)$ is the Dirac's *delta-function.* Then it is well known result that

$$u(x)=\int_A^B g(x,t)f(t)dt \quad \cdots\cdots\cdots\cdots\cdots\cdots (4)$$

is the solution of the differential equation

$$Lu(x)=f(x) \quad \cdots\cdots\cdots\cdots\cdots\cdots (5)$$

satisfying the conditions (2).

Similarly when a linear homogeneous difference equation

$$\bar{L}u(x)=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots (6)$$
$$A\leqq x\leqq B$$

with linear homogeneous boundary conditions

$$B_1(u)=0,\ B_2(u)=0,\ B_3(u)=0,\cdots\cdots, \quad \cdots\cdots\cdots\cdots (7)$$

is given, we can consider the Green's function of the difference operator $\bar{L}$ with boundary conditions (7).

In the following, we supposed that the interval of the independent variable x is $[A,B]$ and that given functions $p(x), q(x), r(x), f(x)$, etc. are one valued functions of x and bounded. Under these supposition, we considered step-function $\bar{H}(x)$, delta-

function $\delta(x)$ etc. as they are considered in the infinitesimal calculus, but there is some difference in definition between the two.

In this paper, Green's functions for difference operators and accompanied difference equations were discussed, especially for the second order linear difference operator, and few examples were given.

In our discussion, we supposed that the difference of independent variable is one, hence

$$\Delta f(x)=f(x+1)-f(x) \text{ etc.,}$$

and also supposed that A, B are both integers and $x=[x]+\varepsilon$ where $0\leqq\varepsilon<1$ and $[x]=$ integer.

§ 2. Step-Function and Delta-Function

We define step-function $\bar{H}(x)$, delta-function $\bar{\delta}(x)$ and a function $K(x)$ as follows

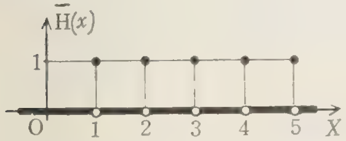

Fig.1

$$\left.\begin{array}{l}\bar{H}(x)=1\cdots\text{for } x=\text{positive integer}\\ \bar{H}(x)=0\cdots\text{for } x\neq\text{positive integer}\end{array}\right\} \cdots\cdots\cdots\cdots\cdots (8)$$

$$\left.\begin{array}{l}\bar{\delta}(x)=0\cdots\text{for } x\neq 0\\ \bar{\delta}(0)=1\end{array}\right\} \cdots\cdots\cdots\cdots\cdots (9)$$

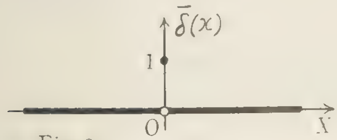

Fig.2

$$\left.\begin{array}{l}K(x)=x\cdots\text{for } x=\text{positive integer}\\ K(x)=1\cdots\text{for } x\neq\text{positive integer}\\ \qquad\qquad\qquad\text{greater than one}\end{array}\right\} \cdots\cdots\cdots\cdots\cdots (10)$$

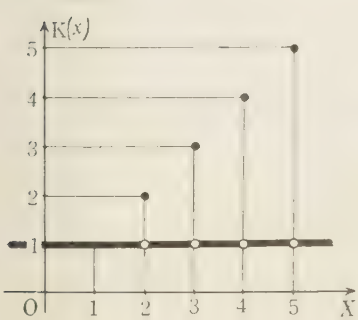

Fig.3

Then, it is evident that there exist following relations

$$\bar{H}(x)=\Delta K(x),\ \bar{\delta}(x)=\Delta\bar{H}(x)=\Delta^2K(x) \cdots\cdots (11)$$

$$f(x)\bar{\delta}(x-t)=f(t)\bar{\delta}(x-t) \cdots\cdots\cdots\cdots\cdots (12)$$

$$\int_{A+\varepsilon}^{B+\varepsilon} f(t)\bar{\delta}(x-t)\Delta t=f(x) \cdots\cdots\cdots\cdots\cdots (13)$$

where $f(x)$ is any function.

§ 3. Green's Function for a Linear Homogeneous Difference Operator

Let S be a space consisted of all functions $u(x)$ which satisfy linear homogeneous conditions $B_1(u)=0$, $B_2(u)=0, \cdots$, and let L be a linear homogeneous difference

operator, and we define the Green's function of L such that

$$\bar{g}(x,t)\in S,\ \bar{L}\bar{g}(x,t)=\bar{\delta}(x-t) \quad \cdots\cdots (14).$$

Then putting to

$$u(x)=\mathop{\mathfrak{S}}_{A+\varepsilon}^{B+\varepsilon}\bar{g}(x,t)f(t)\Delta t \quad \cdots\cdots (15)$$

we get

$$\bar{L}u(x)=\mathop{\mathfrak{S}}_{A+\varepsilon}^{B+\varepsilon}\bar{L}\bar{g}(x,t)f(t)\Delta t=\mathop{\mathfrak{S}}_{A+\varepsilon}^{B+\varepsilon}\bar{\delta}(x-t)f(t)\Delta t=f(x),$$

and since any function $u(x)$ in S must satisfies

$$B_1(u)=0,\ B_2(u)=0,\ B_3(u)=0,\cdots\cdots, \quad \cdots\cdots (16)$$

$g(x,t)$ must satisfies

$$B_1(\bar{g})=0,\ B_2(\bar{g})=0,\ B_3(\bar{g})=0,\cdots\cdots,$$

therefore, for $u(x)$ expressed by (15), we get

$$B_1(u)=\mathop{\mathfrak{S}}_{A+\varepsilon}^{B+\varepsilon}B_1(\bar{g})f(t)\Delta t=0,$$

Similarly $B_2(u)=0,\ B_3(u)=0,\cdots\cdots\cdots\cdots,$
this shows that $u(x)$ expressed by (15) satisfies the conditions (16).

Accordingly, $u(x)$ expressed by (15) is the solution of the difference equation

$$\bar{L}u(x)=f(x) \quad \cdots\cdots (17)$$

under the conditions (16).

§4. Green's Function for Second Order Linear Homogeneous Difference Operator

Consider the Green's function of a difference operator

$$\bar{L}u(x)=p(x)u(x+2)+q(x)u(x+1)+r(x)u(x) \quad \cdots\cdots (18)$$

with linear homogeneous boundary conditions

$$B_1(u)=0,\ B_2(u)=0 \quad \cdots\cdots (19)$$

and we suppose that $p(x), r(x)$ never vanishes*.

At first let $w_1(x)$ and $w_2(x)$ be a set of fundamental solutions of

* The condition $p(x)\neq 0$ is necessary since if $p(x)\equiv 0$, then (20) turns into a first order difference equation, or if there exists a value x_0 such that $p(x_0)=0$, the point corresponding to x_0 generally may be a singular point. Similarly $r(x)\neq 0$ is necessary.

$$Lu(x)=0 \quad \cdots\cdots (20)$$

and we consider to find a particular solution of

$$L\bar{g}(x,t)=\bar{\delta}(x-t) \quad \cdots\cdots (21)$$

with no condition.

As well known, we can solve this equation by *Lagrange*'s method, that is, put

$$\bar{y}(x,t)=\bar{\alpha}_1(x,t)w_1(x)+\bar{\alpha}_2(x,t)w_2(x) \quad (22)$$

$$\bar{y}(x+1,t)=\bar{\alpha}_1(x,t)w_1(x+1)+\bar{\alpha}_2(x,t)w_2(x+1) \quad (23)$$

and determine $\bar{\alpha}_1(x,t)$ and $\bar{\alpha}_2(x,t)$ so as to be that $\bar{y}(x,t)$ expressed by (22) becomes a particular solution of (21). Actually, from (22) we get

$$\bar{y}(x+1,t)=\bar{\alpha}_1(x+1,t)w_1(x+1)+\bar{\alpha}_2(x+1,t)w_2(x+1) \quad (22)'$$

and (22)$'$-(23) gives

$$w_1(x+1)\Delta\bar{\alpha}_1(x,t)+w_2(x+1)\Delta\bar{\alpha}_2(x,t)=0 \quad \cdots\cdots (24).$$

Next, from (23) we get

$$\bar{y}(x+2,t)=\bar{\alpha}_1(x+1,t)w_1(x+2)+\bar{\alpha}_2(x+1,t)w_2(x+2) \quad (23)'$$

and substituting (22), (23) and (23)$'$ into (21) and considering that $Lw_1(x)=0$ and $Lw_2(x)=0$, we get

$$p(x)w_1(x+2)\Delta\bar{\alpha}_1(x,t)+p(x)w_2(x+2)\Delta\bar{\alpha}_2(x,t)=\bar{\delta}(x-t) \quad \cdots\cdots (25)$$

Therefore if

$$D(x)=p(x)\begin{vmatrix} w_1(x+1) & w_2(x+1) \\ w_1(x+2) & w_2(x+2) \end{vmatrix} \neq 0 \quad \cdots\cdots (26),$$

we can solve (24) and (25) with respect to $\Delta\bar{\alpha}_1(x,t)$ and $\Delta\bar{\alpha}_2(x,t)$ simultaneously, and get

$$\Delta\bar{\alpha}_1(x,t)=-\frac{w_2(t+1)}{D(t)}\bar{\delta}(x-t),\ \Delta\bar{\alpha}_2(x,t)=\frac{w_1(t+1)}{D(t)}\bar{\delta}(x-t),$$

and summation of these expressions gives

$$\bar{\alpha}_1(x,t)=-\frac{w_2(t+1)}{D(t)}\bar{H}(x-t),\ \bar{\alpha}_2(x,t)=\frac{w_1(t+1)}{D(t)}\bar{H}(x-t) \quad \cdots\cdots (27),$$

then we get

$$\bar{y}(x,t)=\frac{1}{D(t)}\{w_2(x)w_1(t+1)-w_1(x)w_2(t+1)\}\bar{H}(x-t) \quad \cdots\cdots (28)$$

as a particular solution of (21). Therefore

$$\bar{g}(x,t)=\bar{y}(x,t)+\bar{C}_1(\varepsilon)w_1(x)+\bar{C}_2(\varepsilon)w_2(x) \quad\cdots\cdots (29)$$

is the general solution of (21), where $C_1(\varepsilon)$ and $C_2(\varepsilon)$*
are arbitrary periodic functions of x with their period 1, and they may be determined by the boundary conditions $B_1(\bar{g})=0$, $B_2(\bar{g})=0$.

If these boundary conditions are linear homogeneous, then we get

$$\left.\begin{aligned} B_1(w_1)C_1(\varepsilon)+B_1(w_2)C_2(\varepsilon)+B_1(\bar{y})=B_1(\bar{g})=0\\ B_2(w_1)C_1(\varepsilon)+B_2(w_2)C_2(\varepsilon)+B_2(y)=B_2(g)=0\end{aligned}\right\} \quad\cdots\cdots (30).$$

Hence if

$$\tilde{D}=\begin{vmatrix} B_1(w_1) & B_1(w_2)\\ B_2(w_1) & B_2(w_2)\end{vmatrix}\neq 0 \quad\cdots\cdots (31)$$

then $C_1(\varepsilon)$ and $C_2(\varepsilon)$ can be determined by solving (30).

If $\tilde{D}=0$, either $B_1(w_2)=B_2(w_2)=0$ which implies that w_2 is an eigenfunction of the operator L corresponding to the eigenvalue zero, or there exists a constant k such that

$$B_1(w_1)+kB_1(w_2)=0 \text{ and } B_2(w_1)+kB_2(w_2)=0$$

that is

$$B_1(w_1+kw_2)=0 \text{ and } B_2(w_1+kw_2)=0$$

but $w_1(x)$ and $w_2(x)$ are linearly independent, therefore, w_1+kw_2 never vanishes and also $\bar{L}(w_1+kw_2)=0$ hence w_1+kw_2 is an eigenfunction of the operator L corresponding to the eigenvalue zero.

Therefore if there is no eigenfunction corresponding to the eigenvalue zero, the Green's function exists and is given by (29) where $C_1(\varepsilon)$ and $C_2(\varepsilon)$ are found by solving (30).

§ 5. Solution of Second Order Linear Difference Equation under Linear Non-homogeneous Boundary conditions.

In this section, we consider the solution of

$$Lu(x)=p(x)u(x+2)+q(x)u(x+1)+r(x)u(x)=f(x) \quad\cdots\cdots (18)$$

under the conditions

$$B_1(u)=b(\varepsilon),\ B_2(u)=c(\varepsilon) \quad\cdots\cdots (32)$$

where $B_1(u)$ and $B_2(u)$ are linear homogeneous with respect to $u(x)$, and $b(\varepsilon)$, $c(\varepsilon)$ are given functions of ε.

At first, we can find the solution $u_1(x)$ of the equations

* We have $\varphi(x)\equiv\varphi(\varepsilon)$ for any periodic function $\varphi(x)$ with period 1, where $x=[x]+\varepsilon$.

$$Lu_1(x)=f(x),\ B_1(u_1)=0,\ B_2(u_1)=0 \quad \cdots\cdots\cdots (33)$$

by the method discussed in the preceding section.

Next, let $u_2(x)$ be the solution of

$$Lu_2(x)=0,\ B_1(u_2)=b(\varepsilon),\ B_2(u_2)=c(\varepsilon) \quad \cdots\cdots\cdots (34),$$

then we can easily find $u_2(x)$ as follows.

Since $w_1(x)$, $w_2(x)$ are a set of fundamental solutions of $\bar{L}u(x)=0$, we can put

$$u_2(x)=\alpha_1(\varepsilon)w_1(x)+\alpha_2(\varepsilon)w_2(x)$$

and substituting this formula to the boundary conditions $B_1(u_2)=b(\varepsilon)$ and $B_2(u_2)=c(\varepsilon)$, we get

$$\left.\begin{aligned} B_1(w_1)\alpha_1(\varepsilon)+B_1(w_2)\alpha_2(\varepsilon)=b(\varepsilon)\\ B_2(w_1)\alpha_1(\varepsilon)+B_2(w_2)\alpha_2(\varepsilon)=c(\varepsilon)\end{aligned}\right\} \quad \cdots\cdots\cdots (35).$$

Therefore if

$$\tilde{D}=\begin{vmatrix} B_1(w_1) & B_1(w_2) \\ B_2(w_1) & B_2(w_2) \end{vmatrix} \neq 0 \quad \cdots\cdots\cdots (31),$$

by solving (35) with respect to $\alpha_1(\varepsilon)$ and $\alpha_2(\varepsilon)$ simultaneously we obtain

$$\alpha_1(\varepsilon)=\frac{1}{\tilde{D}}\begin{vmatrix} b(\varepsilon) & B_1(w_2) \\ c(\varepsilon) & B_2(w_2) \end{vmatrix},\quad \alpha_2(\varepsilon)=\frac{1}{D}\begin{vmatrix} B_1(w_1) & b(\varepsilon) \\ B_2(w_1) & c(\varepsilon) \end{vmatrix} \quad \cdots\cdots\cdots (36)$$

and

$$\begin{aligned} u_2(x)=\frac{1}{\tilde{D}}\,[\{B_2(w_2)b(\varepsilon)-B_1(w_2)c(\varepsilon)\}w_1(x) \\ +\{B_1(w_1)c(\varepsilon)-B_2(w_1)b(\varepsilon)\}w_2(x)] \quad \cdots\cdots\cdots (37).\end{aligned}$$

Finally put

$$u(x)=u_1(x)+u_2(x) \quad \cdots\cdots\cdots (38)$$

then we have

$$\begin{aligned} &\bar{L}u(x)=\bar{L}u_1(x)+\bar{L}u_2(x)=f(x)\\ &B_1(u)=B_1(u_1)+B_1(u_2)=b(\varepsilon)\\ &B_2(u)=B_2(u_1)+B_2(u_2)=c(\varepsilon)\end{aligned}$$

therefore $u(x)$ given by (38) is our required solution.

It is evident that the above method can be applied to not only second order linear difference equation but also any order linear difference equation.

§ 6. Examples

Example 1. Find the solution of

$$\Delta^2 u(x) = f(x) \quad \cdots\cdots \quad (\mathrm{i})$$

under the conditions

$$B_1(u) = u(A+\varepsilon) = b(\varepsilon),\ B_2(u) = u(B+\varepsilon) = c(\varepsilon) \quad \cdots\cdots \quad (\mathrm{ii}).$$

Solution:-

At first $w_1(x)=1$ and $w_2(x)=x$ are a set of fundamental solutions of $\Delta^2 u(x)=0$, and

$$D(x) = w_1(x+1)w_2(x+2) - w_1(x+2)w_2(x+1) = 1 \neq 0$$

$$\therefore \qquad \bar{y}(x,t) = (x-t-1)\bar{H}(x-t) \quad \cdots\cdots \quad (\mathrm{iii}),$$

moreover

$$B_1(w_1)=1,\ B_1(w_2)=A+\varepsilon,\ B_1(y)=(A+\varepsilon-t-1)\bar{H}(A+\varepsilon-t)=0$$

$$B_2(w_1)=1,\ B_2(w_2)=B+\varepsilon,\ B_2(y)=(B+\varepsilon-t-1)\bar{H}(B+\varepsilon-t)$$

where $A \leq t \leq B$

$$\therefore \qquad \tilde{D} = B+\varepsilon-(A+\varepsilon) = B-A \neq 0$$

therefore from (30) we get

$$\left.\begin{aligned} \bar{c}_1(\varepsilon) &= \frac{1}{B-A}(A+\varepsilon)(B+\varepsilon-t-1)\bar{H}(B+\varepsilon-t) \\ \bar{c}_2(\varepsilon) &= \frac{1}{B-A}(B+\varepsilon-t-1)\bar{H}(B+\varepsilon-t) \end{aligned}\right\} \quad \cdots\cdots \quad (\mathrm{iv}).$$

Substituting (iii) and (iv) into (29), we obtain

$$\bar{g}(x,t) = (x-t-1)\bar{H}(x-t) + \frac{1}{B-A}(B+\varepsilon-t-1)(A+\varepsilon-x)\bar{H}(B+\varepsilon-t) \quad \cdots\cdots \quad (\mathrm{v})$$

and

$$u_1(x) = \mathop{S}_{A+\varepsilon}^{B+\varepsilon} \left\{(x-t-1)\bar{H}(x-t) + \frac{1}{B-A}(B+\varepsilon-t-1)(A+\varepsilon-x)\bar{H}(B+\varepsilon-t)\right\} f(t)\Delta t \cdots (\mathrm{vi})$$

is the solution of $Lu_1(x)=f(x)$, $B_1(u_1)=0$ and $B_2(u_1)=0$.
We can easily rewrite the right hand side of (vi) as follows

$$u_1(x) = \mathop{S}_{A+\varepsilon}^{x} (x-t-1)f(t)\Delta t + \frac{A+\varepsilon-x}{B-A} \mathop{S}_{A+\varepsilon}^{B+\varepsilon} (B+\varepsilon-t-1)f(t)\Delta t \quad \cdots\cdots \quad (\mathrm{vi}').$$

Next from

$$\left.\begin{array}{l}\alpha_1(\varepsilon)+(A+\varepsilon)\alpha_2(\varepsilon)=b(\varepsilon)\\ \alpha_1(\varepsilon)+(B+\varepsilon)\alpha_2(\varepsilon)=c(\varepsilon)\end{array}\right\} \quad \cdots\cdots\cdots \quad \text{(vii)}$$

we get

$$\left.\begin{array}{l}\alpha_1(\varepsilon)=\dfrac{1}{B-A}\{(B+\varepsilon)b(\varepsilon)-(A+\varepsilon)c(\varepsilon)\}\\ \alpha_2(\varepsilon)=\dfrac{1}{B-A}\{c(\varepsilon)-b(\varepsilon)\}\end{array}\right\} \quad \cdots\cdots\cdots \quad \text{(viii)}$$

and

$$u_2(x)=\frac{1}{B-A}[(B+\varepsilon)b(\varepsilon)-(A+\varepsilon)c(\varepsilon)+\{c(\varepsilon)-b(\varepsilon)\}x] \quad \cdots\cdots\cdots \quad \text{(ix)}$$

is the solution of $\bar{L}u_2(x)=0$ and $B_1(u_2)=b(\varepsilon)$, $B_2(u_2)=c(\varepsilon)$.
Accordingly

$$u(x)=\mathop{\mathfrak{S}}_{A+\varepsilon}^{x}(x-t-1)f(t)\Delta t+\frac{A+\varepsilon-x}{B-A}\mathop{\mathfrak{S}}_{A+\varepsilon}^{B+\varepsilon}(B+\varepsilon-t-1)f(t)\Delta t$$
$$+\frac{1}{B-A}[(B+\varepsilon)b(\varepsilon)-(A+\varepsilon)c(\varepsilon)+\{c(\varepsilon)-b(\varepsilon)\}x] \quad \cdots\cdots\cdots \quad \text{(x)}$$

is the required solution.

Example 2. Find the solution of

$$\bar{L}u(x)=\Delta^2u(x)+k^2u(x)=u(x+2)-2u(x+1)+\rho^2u(x)=f(x) \quad \cdots\cdots\cdots \quad \text{(i)}$$

under the conditions

$$B_1(u)=u(A+\varepsilon)=b(\varepsilon),\ B_2(u)=u(B+\varepsilon)=c(\varepsilon) \quad \cdots\cdots\cdots \quad \text{(ii)}$$

where $\quad k\geqq 0,\ \rho=\sqrt{1+k^2}$.

Solution:-

At first

$$w_1(x)=\rho^x\cos\theta x,\ w_2(x)=\rho^x\frac{\sin\theta x}{\theta} \quad \cdots\cdots\cdots \quad \text{(iii)}$$

are a set of fundamental solutions of $\bar{L}u(x)=0$,
where $\sin\theta=k/\rho$, $\cos\theta=1/\rho$, and

$$D(x)=\begin{vmatrix}\rho^{x+1}\cos\theta(x+1) & \rho^{x+1}\dfrac{\sin\theta(x+1)}{\theta}\\ \rho^{x+2}\cos\theta(x+2) & \rho^{x+2}\dfrac{\sin\theta(x+2)}{\theta}\end{vmatrix}=\frac{\sin\theta}{\theta}\rho^{2x+3}\neq 0 \quad \cdots\cdots\cdots \quad \text{(iv)},$$

hence we get

$$y(x,t)=\rho^{x-t-2}\frac{\sin\theta(x-t-1)}{\sin\theta}H(x-t) \quad\cdots\cdots\cdots\cdots\cdots\cdots \text{(v)}$$

$$B_1(w_1)=\rho^{A+\varepsilon}\cos\theta(A+\varepsilon),\ B_1(w_2)=\rho^{A+\varepsilon}\frac{\sin\theta(A+\varepsilon)}{\theta}$$

$$B_2(w_1)=\rho^{B+\varepsilon}\cos\theta(B+\varepsilon),\ B_2(w_2)=\rho^{B+\varepsilon}\frac{\sin\theta(B+\varepsilon)}{\theta}$$

$$B_1(y)=\rho^{A+\varepsilon-t-2}\frac{\sin\theta(A+\varepsilon-t-1)}{\sin\theta}\bar{H}(A+\varepsilon-t)=0$$

$$B_2(y)=\rho^{B+\varepsilon-t-2}\frac{\sin\theta(B+\varepsilon-t-1)}{\sin\theta}\bar{H}(B+\varepsilon-t)$$

and

$$\tilde{D}=\begin{vmatrix}\rho^{A+\varepsilon}\cos\theta(A+\varepsilon) & \rho^{A+\varepsilon}\dfrac{\sin\theta(A+\varepsilon)}{\theta}\\ \rho^{B+\varepsilon}\cos\theta(B+\varepsilon) & \rho^{B+\varepsilon}\dfrac{\sin\theta(B+\varepsilon)}{\theta}\end{vmatrix}=\frac{\sin\theta(B-A)}{\theta}\cdot\rho^{A+B+2\varepsilon}.$$

Therefore if $\theta(B-A)\neq n\pi(n=\text{integer})$, then from (30) we get

$$c_1(\varepsilon)=\rho^{-t-2}\frac{\sin\theta(A+\varepsilon)}{\sin\theta(B-A)}\frac{\sin\theta(B+\varepsilon-t-1)}{\sin\theta}H(B+\varepsilon-t)$$

$$\bar{c}_2(\varepsilon)=-\rho^{-t-2}\frac{\theta}{\sin\theta(B-A)}\frac{\sin\theta(B+\varepsilon-t-1)}{\sin\theta}\cos\theta(A+\varepsilon)\bar{H}(B+\varepsilon-t)$$

and

$$\bar{g}(x,t)=\rho^{x-t-2}\left[\frac{\sin\theta(x-t-1)}{\sin\theta}\bar{H}(x-t)\right.$$
$$\left.-\frac{\sin\theta(x-A-\varepsilon)}{\sin\theta(B-A)}\frac{\sin\theta(B+\varepsilon-t-1)}{\sin\theta}\bar{H}(B+\varepsilon-t)\right] \quad\cdots\cdots\cdots \text{(vi)},$$

therefore

$$u_1(x)=\frac{1}{\sin\theta}\left[\mathop{\mathfrak{S}}_{A+\varepsilon}^{x}\rho^{x-t-2}\sin\theta(x-t-1)f(t)\Delta t\right.$$
$$\left.-\frac{\sin\theta(x-A-\varepsilon)}{\sin\theta(B-A)}\mathop{\mathfrak{S}}_{A+\varepsilon}^{B+\varepsilon}\rho^{x-t-2}\sin\theta(B+\varepsilon-t-1)f(t)\Delta t\right] \cdots \text{(vii)}.$$

Next, from (35) we get

$$\alpha_1(\varepsilon)=\frac{1}{\rho^{A+B+\varepsilon}\sin\theta(B-A)}\{\rho^B b(\varepsilon)\sin\theta(B+\varepsilon)-\rho^A c(\varepsilon)\sin\theta(A+\varepsilon)\}$$

$$\alpha_2(\varepsilon)=\frac{1}{\rho^{A+B+\varepsilon}\sin\theta(B-A)}\{\rho^A c(\varepsilon)\cos\theta(A+\varepsilon)-\rho^B b(\varepsilon)\cos\theta(B+\varepsilon)\},$$

and

$$u_2(x)=\frac{\rho^x}{\rho^{A+B+\varepsilon}\sin\theta(B-A)}\{\rho^B b(\varepsilon)\sin\theta(B+\varepsilon-x)+\rho^A c(\varepsilon)\sin\theta(x-A-\varepsilon)\}\cdots\text{(viii)}.$$

Accordingly

$$u(x)=u_1(x)+u_2(x)\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\text{(ix)}$$

is the required solution, where $u_1(x)$ and $u_2(x)$ are given by (vii) and (viii) respectively.

If we consider the case in which $k\to 0$, then we have $\rho\to 1$ and $\theta\to 0$, hence

$$\lim_{k\to 0}\frac{\sin\theta(x-t-1)}{\sin\theta}=x-t-1$$

$$\lim_{k\to 0}\frac{\sin\theta(B+\varepsilon-t-1)\sin\theta(x-A-\varepsilon)}{\sin\theta(B-A)\sin\theta}=\frac{(x-A-\varepsilon)(B+\varepsilon-t-1)}{B-A}$$

therefore

$$\lim_{k\to 0}u_1(x)=\mathop{S}\limits_{A+\varepsilon}^{x}(x-t-1)f(t)\Delta t-\frac{x-A-\varepsilon}{B-A}\mathop{S}\limits_{A+\varepsilon}^{B+\varepsilon}(B+\varepsilon-t-1)f(t)\Delta t\ \cdots\cdots\text{(vii)}',$$

similarly

$$\lim_{k\to 0}u_2(x)=\frac{1}{B-A}\{(B+\varepsilon-x)b(\varepsilon)+(x-A-\varepsilon)c(\varepsilon)\}\ \cdots\cdots\text{(viii)}',$$

and

$$\lim_{k\to 0}u(x)=\mathop{S}\limits_{A+\varepsilon}^{x}(x-t-1)f(t)\Delta t+\frac{A+\varepsilon-x}{B-A}\mathop{S}\limits_{A+\varepsilon}^{B+\varepsilon}(B+\varepsilon-t-1)f(t)\Delta t$$
$$+\frac{1}{B-A}\{(B+\varepsilon-x)b(\varepsilon)+(x-A-\varepsilon)c(\varepsilon)\}\ \cdots\cdots\text{(ix)}'$$

and this coincides with the result of example 1.

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NUMERICAL INTEGRATIONS FOR THE HIGHLY DEGENERATE ISOTHERMAL CORES OF COLLAPSED STARS

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(Received January 31, 1962)

Abstract

In order to investigate the internal structure of collapsed stars, the numerical integrations for highly degenerate isothermal cores were carried out in the four cases of models with the values of $\psi_c=70$, 100, 150 and 200, where ψ_c indicates the degree of degeneracy at the centre of the star.

1. Introduction

At present it is considered that in the interior of the usual white dwarf star the electron gas is in a completely degenerate state (as the first approximation), that is $\psi_c=+\infty$. This may, however, be not the case in the interior of the star at pre-white-dwarf stage, that is, the value of ψ_c is finite. In order to study the internal structure of such a collapsed star, we should obtain the solution of highly degenerate isothermal gas sphere.

2. Basic Equations

The fundamental equations (hydrostatic equilibrium conditions) are

$$\frac{dP}{dr}=-G\frac{M_r\rho}{r^2}, \quad \frac{dM_r}{dr}=4\pi r^2\rho. \qquad (1)$$

The equation of state can be written in the parametric form

$$\left.\begin{aligned} P&=\frac{8\pi}{3h^3}(2mkT)^{3/2}kTF_{3/2}(\psi)+\frac{\rho kT}{\mu_n H},\\ \rho&=\mu_e H\frac{4\pi}{h^3}(2mkT)^{3/2}F_{1/2}(\psi),\\ F_n(\psi)&=\int_0^\infty\frac{u^n du}{e^{u-\psi}+1}, \end{aligned}\right\} \qquad (2)$$

where μ_e and μ_n are the mean molecular weights of electrons and ions, respectively. Introducing the equations (2) and the following transformations into eq. (1)

$$\left.\begin{aligned} r&=(h^{3/2}/4\pi\mu_e H)(2mkT)^{-1/4}(2mG)^{-1/2}\xi\\ M_r&=(h^{3/2}/4\pi\mu_e^2H^2)(2mkT)^{3/4}(2mG)^{-3/2}\varphi, \end{aligned}\right\} \qquad (3)$$

we can obtain

$$\frac{d\psi}{d\xi}=-\frac{\varphi}{\xi^2}\frac{1}{1+\delta},\quad \frac{d\varphi}{d\xi}=\xi^2F_{1/2}, \quad\cdots\cdots\cdots\cdots\cdots \tag{4}$$

with

$$\delta=\frac{\mu_e}{\mu_n}\frac{1}{F_{1/2}}\frac{dF_{1/2}}{d\psi}. \quad\cdots\cdots\cdots\cdots\cdots \tag{5}$$

For the convenience of calculations, if we introduce the logarithmic variables

$$y=\log_{10}\xi,\quad \sigma=\log_{10}\varphi,\quad \eta=\log_{10}F_{1/2}-\zeta \quad\cdots\cdots\cdots\cdots\cdots \tag{6}$$

with

$$\zeta=-\log_{10}[(1+\delta)\log_{10}e],$$

we obtain

$$\log_{10}\left(-\frac{dy}{d\psi}\right)=y-\sigma-\zeta,\quad \log_{10}\left(-\frac{d\sigma}{d\psi}\right)=4y+\eta-2\sigma. \quad\cdots\cdots\cdots\cdots\cdots \tag{7}$$

These eqns. for y and σ are integrated with the boundary condition $\xi=0$ and $\varphi=0$ at $\psi=\psi_c$ (at the centre of the star).
The invariant functions U and V are given by

$$U=\frac{d\log M_r}{d\log r}=\frac{d\sigma}{dy} \quad\cdots\cdots\cdots\cdots\cdots \tag{8}$$

$$V=-\frac{d\log P}{d\log r}=-\frac{d\lambda}{dy} \quad \text{with } \lambda=\log_{10}\left(\frac{2}{3}F_{3/2}+\frac{\mu_e}{\mu_n}F_{1/2}\right). \quad\cdots\cdots\cdots \tag{9}$$

3. Tabulation of Integrations

The values of ψ, y, σ, U and V are tabulated in four cases of $\psi_c=70$, 100, 150 and 200. In these tables all actual numerical calculations have been carried out in the case of $\mu_n/\mu_e=2$, which is exactly valid for the fully ionized gas of pure helium.

In the cases of $\psi_c=70$ and 100, integrations were directly started near the centre with their starting values. But in the cases of $\psi_c=150$ and 200, integrations were started from $\psi=75$ at $\xi=0.696\,11$ and $\psi=100$ at $\xi=0.647\,81$, respectively. These starting values can be determind, by the interpolation, from the table of Emden function with polytropic index $n=1.5$, which is the solution of eq.

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\theta}{dx}\right)=-\theta^{3/2} \quad\cdots\cdots\cdots\cdots\cdots \tag{10}$$

with a condition $\theta=1$ at $x=0$, because in the case of $\delta\ll 1$ eqns. (4) can be easily reduced to eq. (10). For the present cases, it may be shown that we make no error at the fifth digid by using such approximations.

The integrations were performed with much shorter length of step and with one more digit than are given in the tabulations.

This work has been supported by the Scientific Research Grands of Department of Education.

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(1) $\psi_c = 70$

ψ	log ξ	log φ	log U	log V
70	$-\infty$	$-\infty$		
69	$\bar{1}.098\ 2$	$\bar{1}.403\ 1$		
68	$1.249\ 5$	$\bar{1}.851\ 9$		
67	$\bar{1}.338\ 8$	$0.114\ 0$		
66	$\bar{1}.402\ 7$	$0.299\ 7$		
65	$\bar{1}.452\ 6$	$0.443\ 4$		
60	$\bar{1}.610\ 5$	$0.886\ 9$		
55	$\bar{1}.706\ 5$	$1.142\ 4$		
50	$\bar{1}.777\ 5$	$1.320\ 5$		
45	$\bar{1}.835\ 1$	$1.455\ 6$		
40	$\bar{1}.884\ 6$	$1.562\ 9$		
35	$\bar{1}.928\ 7$	$1.650\ 5$		
30	$\bar{1}.969\ 2$	$1.723\ 0$		
25	$0.007\ 6$	$1.783\ 0$		
20	$0.044\ 6$	$1.832\ 2$	$0.078\ 5$	$0.853\ 1$
15	$0.081\ 4$	$1.871\ 2$	$\bar{1}.963\ 4$	$0.968\ 2$
10	$0.119\ 0$	$1.900\ 1$	$\bar{1}.786\ 4$	$1.109\ 5$
5	$0.159\ 5$	$1.918\ 1$	$\bar{1}.454\ 6$	$1.301\ 5$
4	$0.168\ 3$	$1.920\ 3$	$\bar{1}.345\ 9$	$1.347\ 8$
3	$0.177\ 5$	$1.922\ 1$	$\bar{1}.209\ 9$	$1.395\ 2$
2	$0.187\ 3$	$1.923\ 4$	$\bar{1}.036\ 9$	$1.440\ 6$
1	$0.197\ 7$	$1.924\ 3$	$\bar{2}.813\ 6$	$1.478\ 5$
0	$0.208\ 9$	$1.924\ 8$	$\bar{2}.533\ 0$	$1.502\ 9$
−1	$0.220\ 9$	$1.925\ 1$	$\bar{2}.200\ 9$	$1.511\ 7$
−2	$0.233\ 7$	$1.925\ 3$	$\bar{3}.835\ 1$	$1.508\ 9$

(2) $\psi_c = 100$

ψ	log ξ	log φ	log U	log V
100	$-\infty$	$-\infty$		
99	$\bar{2}.982\ 0$	$\bar{1}.286\ 3$		
98	$\bar{1}.133\ 2$	$\bar{1}.737\ 3$		
97	$\bar{1}.221\ 7$	$\bar{1}.999\ 5$		
96	$\bar{1}.284\ 9$	$0.185\ 3$		
95	$1.334\ 2$	$0.329\ 1$		
90	$\bar{1}.489\ 4$	$0.774\ 4$		
85	$\bar{1}.582\ 7$	$1.032\ 5$		
80	$\bar{1}.650\ 7$	$1.213\ 8$		
75	$\bar{1}.704\ 9$	$1.352\ 6$		

(2) $\psi_c = 100$ (Continued)

ψ	$\log \xi$	$\log \varphi$	$\log U$	$\log V$
70	1.750 6	1.464 6		
65	1.790 5	1.557 7		
60	1.826 1	1.636 7		
55	1.858 9	1.704 9		
50	1.889 2	1.764 3		
45	1.918 1	1.816 3		
40	1.945 4	1.861 9		
35	1.971 8	1.901 8		
30	1.997 6	1.936 3		
25	0.023 2	1.966 0		
20	0.048 8	1.990 5	$\bar{1}$.932 8	1.007 3
15	0.074 8	2.010 0	$\bar{1}$.805 0	1.113 6
10	0.101 9	2.024 2	$\bar{1}$.611 0	1.250 8
5	0.131 1	2.032 7	$\bar{1}$.255 0	1.444 5
3	0.144 0	2.034 5	$\bar{2}$.997 2	1.541 1
2	0.150 9	2.035 0	$\bar{2}$.816 0	1.588 5
1	0.158 3	2.035 4	$\bar{2}$.584 6	1.629 0
0	0.166 2	2.035 6	$\bar{2}$.294 3	1.656 3
−1	0.174 6	2.035 8	$\bar{3}$.951 3	1.668 7
−2	0.183 5	2.035 8	$\bar{3}$.573 8	1.669 7

(3) $\psi_c = 150$

ψ	$\log \xi$	$\log \varphi$	$\log U$	$\log V$
150	$-\infty$	$-\infty$		
75	1.842 7	1.826 5		
70	1.864 8	1.873 3		
65	1.885 8	1.915 2		
60	1.905 9	1.952 7		
55	1.925 3	1.986 5		
50	1.944 2	2.016 9		
45	1.962 6	2.044 1		
40	1.980 8	2.068 3		
35	1.998 8	2.089 8		
30	0.016 8	2.108 4		
25	0.034 9	2.124 1		
20	0.053 3	2.137 2	$\bar{1}$.799 6	1.149 5
15	0.072 1	2.147 4	$\bar{1}$.659 5	1.253 8
10	0.091 7	2.154 7	$\bar{1}$.450 0	1.391 4
5	0.112 8	2.158 8	$\bar{1}$.073 8	1.588 9
3	0.122 0	2.159 6	$\bar{2}$.805 8	1.688 2
2	0.126 9	2.159 9	$\bar{2}$.619 4	1.737 4
1	0.132 2	2.160 1	$\bar{2}$.381 5	1.779 8
0	0.137 7	2.160 2	$\bar{2}$.084 3	1.809 3
−1	0.143 7	2.160 2	$\bar{3}$.734 0	1.824 1

(4) $\psi_c=200$

ψ	$\log \xi$	$\log \varphi$	$\log U$	$\log V$
200	$-\infty$	$-\infty$		
100	$\bar{1}.811\ 5$	1.982 7		
95	$\bar{1}.826\ 0$	2.009 6		
90	$\bar{1}.840\ 1$	2.034 8		
85	$\bar{1}.853\ 9$	2.058 3		
80	$\bar{1}.867\ 5$	2.080 4		
75	1.880 8	2.101 0		
70	$\bar{1}.893\ 9$	2.120 2		
65	$\bar{1}.907\ 5$	2.138 2		
60	$\bar{1}.920\ 3$	2.154 8		
55	$\bar{1}.933\ 1$	2.170 2		
50	$\bar{1}.945\ 8$	2.184 3		
45	$\bar{1}.958\ 6$	2.197 2		
40	$\bar{1}.971\ 4$	2.208 8		
35	$\bar{1}.984\ 2$	2.219 1		
30	$\bar{1}.997\ 3$	2.228 0		
25	0.010 5	2.235 7		
20	0.024 0	2.241 9	$\bar{1}.606\ 9$	1.283 6
15	0.037 8	2.246 7	$\bar{1}.456\ 9$	1.387 4
10	0.052 2	2.250 0	$\bar{1}.235\ 9$	1.526 3
5	0.067 6	2.251 8	$\bar{2}.845\ 0$	1.727 2
3	0.074 3	2.252 2	$\bar{2}.570\ 2$	1.828 5
2	0.077 9	2.252 3	$\bar{2}.379\ 5$	1.878 8
1	0.081 6	2.252 4	$\bar{2}.137\ 7$	1.922 6
0	0.085 6	2.252 4	$\bar{3}.835\ 8$	1.953 7
−1	0.089 9	2.252 4	$\bar{3}.621\ 3$	1.970 1

NOTE ON MALCEV ALGEBRAS

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(Received Aug. 30, 1962)

A Malcev algebra is an anti-commutative algebra defined by the Malcev condition $(xy)(zx)+(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y=0$. In an anti-commutative algebra A, the Jacobi condition $(xy)z+(yz)x+(zx)y=0$ says that a mapping $x\to ax$ is a derivation in A. In this note we show that this situation holds for the Malcev condition with a suitable modification. We next show that any Malcev algebra can be made into a certain subspace of a Lie algebra satisfying some conditions. For this purpose it is important to consider a trilinear composition $[xyz]=x(yz)-y(xz)+(xy)z$ with the original composition xy.

1. Axioms of Malcev algebras. An anti-commutative algebra A over a field Φ is a non-associative algebra satisfying

$$x^2=0 \qquad \text{for all } x\in A. \tag{1.1}$$

A *Malcev algebra* M or a *Moufang-Lie algebra* [2][1] is an anti-commutative algebra satisfying

$$(xy)(zx)+(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y=0 \qquad \text{for all } x,y,z\in M. \tag{1.2}$$

Any Lie algebra is a Malcev algebra. Let C be a Cayley-Dickson algebra with a multiplication xy. An algebra derived from C with a new multiplication $[x,y]=xy-yx$ is a Malcev algebra but not a Lie algebra.

Let M_1 be an anti-commutative algebra defined by the identity:

$$[x,y,zw]=[xyz]w+z[xyw], \tag{1.3}$$

where

$$[xyz]=x(yz)-y(xz)+(xy)z. \tag{1.4}$$

(1.3) shows that a linear mapping $z\to\sum_i[x_iy_iz]$ is a derivation in M_1. Then we have the following

THEOREM 1.1. *In an anti-commutative algebra over a field Φ of characteristic not 2 or 3, the Malcev condition (1.2) is equivalent to (1.3).*

That a Malcev algebra over a field Φ of characteristic different from 2 is an algebra M_1 is proved by Sagle [3, Prop. 8.3]. Hence we prove the converse under

1) Numbers in brackets refer to the references at the end of the paper.

the assumption that the characteristic of Φ is not 2 or 3. In an anti-commutative algebra, put

$$g(x,y,z,w)=[x,y,zw]+[z,w,xy]. \tag{1.5}$$

Clearly $g(x,x,y,z)=g(x,y,z,z)=0$. Also, put

$$J(x,y,z)=(xy)z+(yz)x+(zx)y, \tag{1.6}$$

then J is a skew-symmetric function of its arguments.

LEMMA 1.1. *An algebra M_1 of characteristic not 2 is a Malcev algebra if and only if the function g is skew-symmetric with respect to its variables.*

PROOF. Let M_1 be a Malcev algebra. The identity $g(x,y,z,w)+J(x,y,zw)+J(xy,z,w)=0$ and the result of Kleinfeld [1] imply that g is skew-symmetric with respect to its arguments. Conversely, assume that g is a skew-symmetric function in an algebra M_1. In an anti-commutative algebra it holds the identity: $[x,y,zw]-[xyz]w-z[xyw]=(xy)(zw)+x(z\cdot wy)+w(x\cdot yz)+y(w\cdot zx)+z(y\cdot xw)+g(x,y,z,w)+g(y,w,x,z)$. Hence we have $(xy)(zw)+x(z\cdot wy)+w(x\cdot yz)+y(w\cdot zx)+z(y\cdot xw)=0$[2], which implies (1.2) by putting $w=x$. q. e. d.

Next, by using the identity

$$J(x,y,z)+[xyz]=2(xy)z \tag{1.7}$$

in an anti-commutative algebra and that $z\to[xyz]$ is a derivation in an algebra M_1, we have $J(x,y,zw)=2(xy)(zw)-[xyz]w-z[xyw]$ in M_1, hence by (1.7) we obtain

$$J(x,y,zw)+2J(xy,z,w)=J(x,y,z)w+zJ(x,y,w).$$

In this relation, if we interchange x with z and y with w respectively, we have

$$J(z,w,xy)+2J(zw,x,y)=J(z,w,x)y+xJ(z,w,y).$$

Adding these two relations and using that J is a skew-symmetric function, we have

$$3g(x,y,z,w)+J(x,y,z)w+zJ(x,y,w)+J(z,w,x)y+xJ(z,w,y)=0.$$

Since the characteristic of Φ is not 3, we obtain $g(x,y,y,z)=0$ and g is skew-symmetric with respect to its variables. Hence Theorem 1.1 is proved from Lemma 1.1.

2. Relation between Malcev algebras and Lie algebras. In this section we prove the following

THEOREM 2.1. *For a Malcev algebra M over a field Φ of characteristic not 2, there exists a Lie algebra $\mathfrak{L}$ such that $\mathfrak{L}=M\oplus\mathfrak{D}$ (a vector space direct sum) and $[\mathfrak{D},\mathfrak{D}]\subseteq\mathfrak{D}$, $[M,\mathfrak{D}]\subseteq M$ and where the product xy in M is an M-component of a product $[x,y]$ in $\mathfrak{L}$.*

2) We remark that this relation is equivalent to (1.2) by [3, Prop. 2.21].

PROOF. We have the following identities in M.

$$[xxy]=0, \tag{2.1}$$

$$[xyz]+[yzx]+[zxy]+(xy)z+(yz)x+(zx)y=0, \tag{2.2}$$

and

$$[xy,z,w]+[yz,x,w]+[zx,y,w]=0. \tag{2.3}$$

In fact, put $F(x,y,z)=(xy)(zx)+(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y$, then since the characteristic of Φ is not 2, $F(x+w,y,z)+F(y+w,z,x)+F(z+w,x,y)=0$ implies (2.3). Applying the derivation $D(x,y)$: $z\to[xyz]$ for the ternary product $[zvw]$ we have

$$[xy[zvw]]=[[xyz]vw]+[z[xyv]w]+[zv[xyw]], \tag{2.4}$$

from this

$$[D(x,y),\ D(z,w)]=D([xyz],w)+D(z,[xyw]). \tag{2.5}$$

Let $\mathfrak{D}(M)$ be a vector space over Φ spanned by $\sum_i D(x_i,y_i)$'s, then $\mathfrak{D}(M)$ is a Lie algebra by (2.5). Let $\mathfrak{L}$ be a vector space direct sum $M\oplus\mathfrak{D}(M)$, then the element of $\mathfrak{L}$ is of the form $x+\sum_i D(y_i,z_i)$. A multiplication in $\mathfrak{L}$ is defined as follows:

$$\begin{aligned}&[x+\sum_i D(y_i,z_i),\ u+\sum_i D(v_i,w_i)]\\ &\quad=xu+\sum_i[y_iz_iu]-\sum_i[v_iw_ix]+D(x,u)\\ &\qquad\qquad+\sum_i D([y_iz_iv_i],w_i)+\sum_i D(v_i,[y_iz_iw_i]),\end{aligned}$$

then from (1.1), (1.3), (2.1),$\cdots$, (2.4) $\mathfrak{L}$ is a Lie algebra such that $[M,\mathfrak{D}(M)]\subseteq M$ and xy is an M-component of $[x,y]$ for $x,y\in M$ [6]. Therefore the theorem is proved.

COROLLARY 2.1. *That a Malcev algebra M reduces to a Lie algebra relative to the original composition xy is equivalent to that M is a Lie triple system*[3] *relative to the ternary composition* $[xyz]=x(yz)-y(xz)+(xy)z$.

COROLLARY 2.2. *If a Malcev algebra M satisfies $[MMM]=0$, then M is a Lie algebra.*

REMARK. If M is finite-dimensional, then the Lie algebra $\mathfrak{L}=M\dotplus\mathfrak{D}(M)$ is also finite-dimensional, since $\dim\mathfrak{L}\leqq n(n+1)/2$ where $n=\dim M$. Suppose that a vector space T has a binary composition xy and a ternary composition $[xyz]$ satisfying (1.1), (1.3), (2.1),$\cdots$, (2.4), then T is called a *general Lie triple system* [6]. Therefore a Malcev algebra has a structure of general Lie triple system relative to the com-

3) A Lie triple system is a vector space with a trilinear composition $[xyz]$ satisfying $[xxy]=0$, $[xyz]+[yzx]+[zxy]=0$ and $[xy[zvw]]=[[xyz]vw]+[z[xyv]w]+[zv[xyw]]$. See [5].

positions xy and $[xyz]=x(yz)-y(xz)+(xy)z$.

Let $\mathfrak{L}$ be a Lie algebra over $\boldsymbol{R}$ with a basis $X_1=\frac{\partial}{\partial x}$, $X_2=x\frac{\partial}{\partial x}$, $X_3=x\frac{\partial}{\partial y}$, $X_4=y\frac{\partial}{\partial x}$, $X_5=\frac{\partial}{\partial y}$, $X_6=y\frac{\partial}{\partial y}$ and let M and $\mathfrak{D}$ be the subspaces of $\mathfrak{L}$ with bases X_1, X_2, X_3, X_4 and X_5, X_6 respectively. Then $\mathfrak{L}$ is the vector space direct sum $M\oplus\mathfrak{D}$ and $[\mathfrak{D}, \mathfrak{D}]\subseteqq\mathfrak{D}$, $[M, \mathfrak{D}]\subseteqq M$. For X_1, X_2, X_3, X_4 define

$$X_iX_j=[X_i, X_j]_M,$$
$$[X_iX_jX_k]=[[X_i, X_j]_{\mathfrak{D}}, X_k],$$

where $[X_i, X_j]_M$ and $[X_i, X_j]_{\mathfrak{D}}$ denote the M-component of $[X_i, X_j]$ and the $\mathfrak{D}$-component of $[X_i, X_j]$ respectively. Then, M is a general Lie triple system relative to XY and $[XYZ]$ but not a Malcev algebra relative to XY, in fact for $X=X_1+X_2$, $Y=X_4$, $Z=X_1$ $(XY)(ZX)+(XY\cdot Z)X+(YZ\cdot X)X+(ZX\cdot X)Y=-2X_1\neq 0$.

PROPOSITION 2.1. *In a general Lie triple system T with compositions xy and $[xyz]$, the Malcev condition* (1.2) *is equivalent to the following condition:*

$$[x, y, zx]+[y, zx, x]+[zx, x, y]+[xyz]x+[yzx]x+[zxy]x=0.$$

PROOF. From (2.2) we have the identity: $(xy)(zx)+(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y+[x, y, zx]+[y, zx, x]+[zx, x, y]+[xyz]x+[yzx]x+[zxy]x=0$, which proves the proposition.

Let N be a subalgebra of a Malcev algebra M and let $\mathfrak{D}(N, N)$ be a Lie algebra generated by $\sum_i D(x_i, y_i)$'s, $x_i, y_i\in N$. From Theorem 2.1 the vector space direct sum $N\oplus\mathfrak{D}(N, N)$ becomes a Lie algebra. A subalgebra $\mathfrak{A}$ of a Lie algebra $\mathfrak{L}$ is called to be subinvariant in $\mathfrak{L}$ if there exists a finite sequence of subalgebras $\mathfrak{L}=\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_r=\mathfrak{A}$ such that $\mathfrak{A}_i$ is an ideal in $\mathfrak{A}_{i-1}, i=1, 2, \cdots, r$ [4]. Let N be an ideal of a Malcev algebra M, then the next proposition shows that the Lie algebra $N\oplus\mathfrak{D}(N, N)$ is subinvariant in the Lie algebra $M\oplus\mathfrak{D}(M, M)$.

PROPOSITION 2.2. *Let N be an ideal of a Malcev algebra M. Then, the Lie algebra $N\oplus\mathfrak{D}(N, N)$ is an ideal of the Lie algebra $N\oplus\mathfrak{D}(N, M)$ and $N\oplus\mathfrak{D}(N, M)$ is an ideal of the Lie algebra $M\oplus\mathfrak{D}(M, M)$.*

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昭和37年11月10日 印刷
昭和37年11月14日 発行
非売品

編輯兼発行者　熊本大学理学部
印刷者　佐伯善次
印刷所　佐伯印刷所
熊本市大江町九品寺138
電話④2355・④2958番